Higher-order intersections in low-dimensional topology

Freedman Fest June 2011

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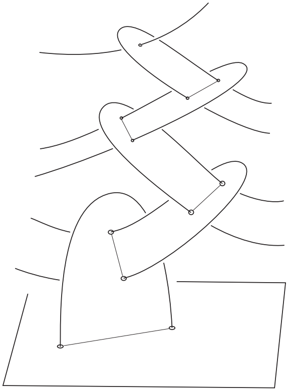
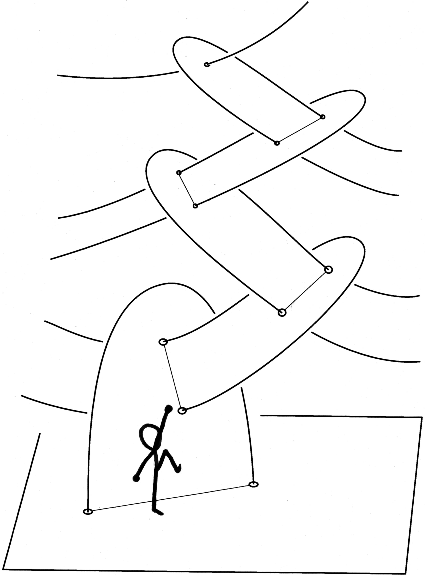
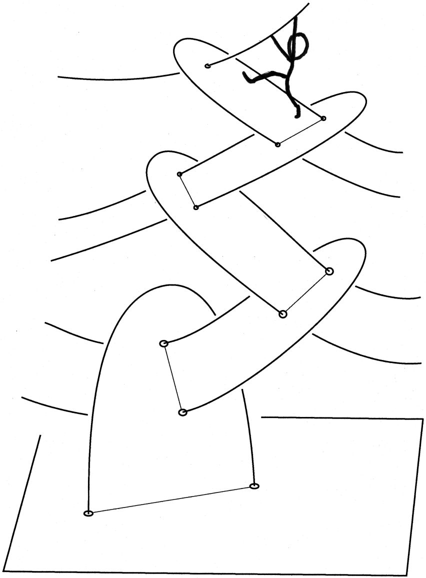
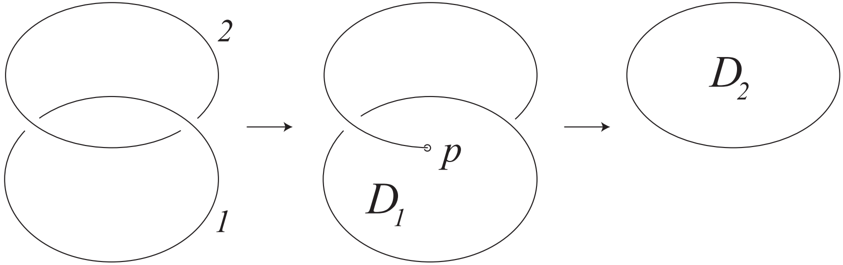






Linking number as an order 0 intersection invariant:

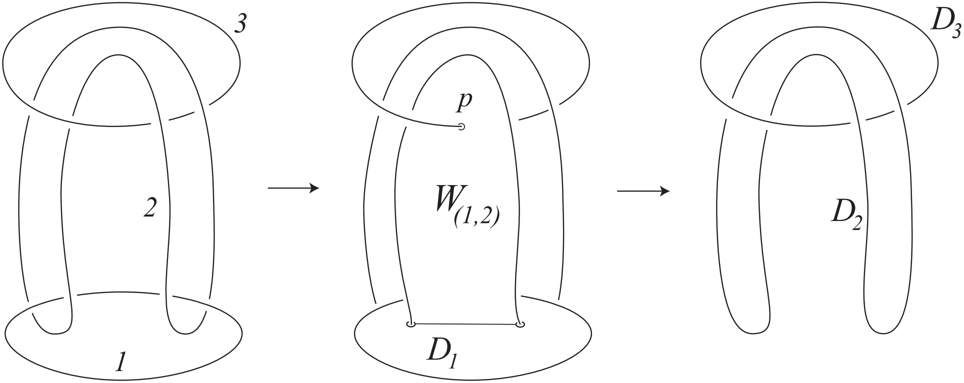
Moving into B^4 from left to right.



$\mathscr{W} := D_1 \cup D_2 \subset B^4$ is an order 0 Whitney tower with intersection invariant

$$\tau_0(\mathscr{W}) = \ell k(\partial \mathscr{W}) \in \mathbb{Z}$$

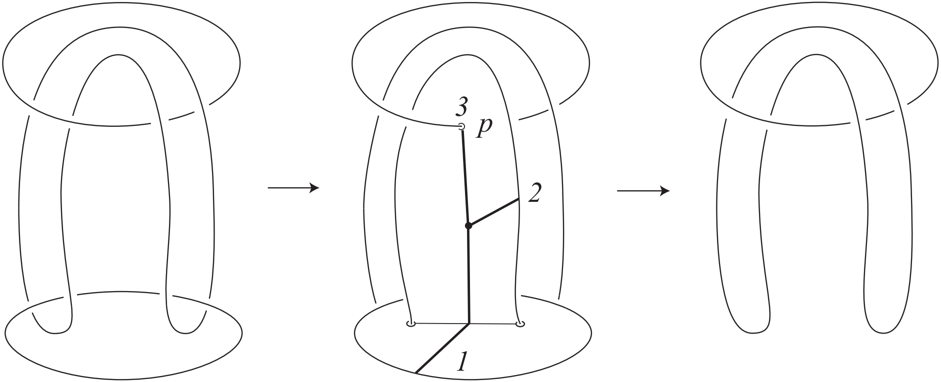
Vanishing order 0 intersections $\rightsquigarrow$ Order 1 Whitney tower $\mathscr{W}$



$\mathscr{W} = D_1 \cup D_2 \cup D_3 \cup W_{(1,2)} \subset B^4$ is an order 1 Whitney tower.
$p \in W_{(1,2)} \cap D_3$ is an order 1 intersection point.

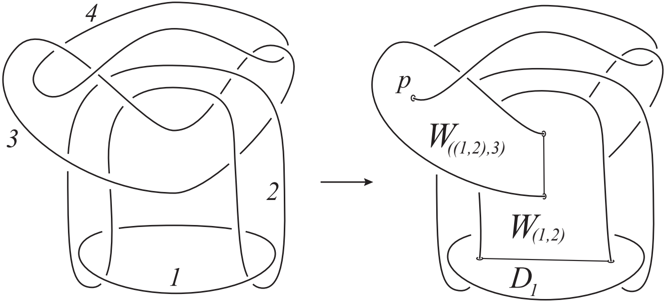
The order 1 intersection invariant τ_1.

Order 1 trees are associated to order 1 intersections.



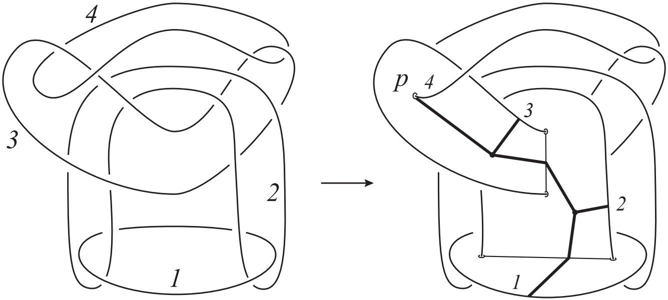
$$\tau_1(\mathscr{W}) = t_p = 3\!\!-\!\!\!<\!\begin{smallmatrix}2\\1\end{smallmatrix} \in \mathscr{T}_1(3) = \frac{\text{order 1 trees}}{\text{relations}}$$

Vanishing order ≤ 1 intersections $\rightsquigarrow$ Order 2 Whitney tower



$$\mathscr{W} = D_1 \cup D_2 \cup D_3 \cup D_4 \cup W_{(1,2)} \cup W_{((1,2),3)} \subset B^4$$

The order 2 intersection invariant τ_2.



$$\tau_2(\mathscr{W}) \;=\; t_p \;=\; \begin{smallmatrix}3\\4\end{smallmatrix} \!>\!\!-\!\!<\! \begin{smallmatrix}2\\1\end{smallmatrix} \in \mathscr{T}_2(4) := \frac{\text{order 2 trees}}{\text{relations}}$$

Order n twisted Whitney towers in B^4.

- Construction Theorem: If $L \subset S^3$ bounds $\mathcal{W} \subset B^4$ with $\tau_n^\infty(\mathcal{W}) = 0$ then L bounds an order $n+1$ twisted Whitney tower.

- Detection Theorem: The first non-vanishing (length $n+2$) Milnor invariants of $L = \partial\mathcal{W}$ can be computed from $\tau_n^\infty(\mathcal{W})$.

Order n twisted Whitney towers in B^4.

<u>Classification Theorem:</u> The sets W_n^∞ of links bounding order n twisted Whitney towers modulo order $n+1$ twisted Whitney tower concordance are finitely generated abelian groups which are classified by Milnor invariants and higher-order Arf invariants.

Specifically,

$$\mathsf{W}_n^\infty \cong \mathbb{Z}^{N_n} \quad \text{for} \quad n \equiv 0,1,3 \mod 4$$

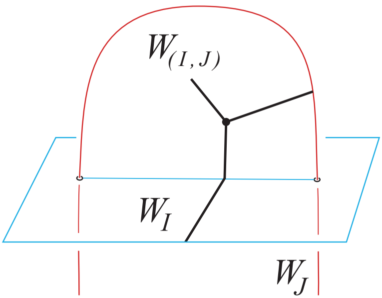
where N_n is the number of independent Milnor invariants;

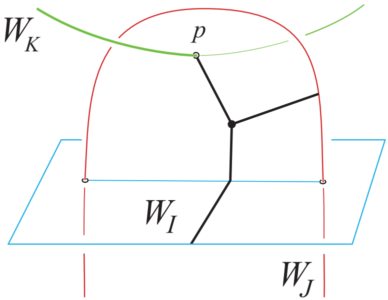
$$\mathsf{W}_{4k-2}^\infty \cong \mathbb{Z}^{N_{4k-2}} \oplus (\mathbb{Z}_2 \otimes \mathscr{L}_k)/?$$

where the torsion quotient is detected by higher-order Arf invariants.

Unitrivalent trees for Whitney disks and intersections

Whitney disks $\rightsquigarrow$ rooted trees. Transverse intersections $\rightsquigarrow$ un-rooted trees.





$$(I,J) \longleftrightarrow \;-\!\!\!<^{J}_{I} \qquad \text{and} \qquad t_p = K\!-\!\!\!<^{J}_{I} =: \langle K,(I,J)\rangle$$

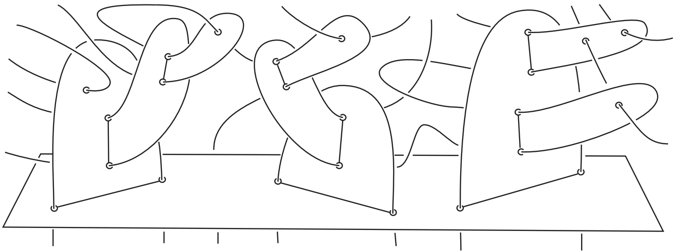
Grading by order = number of trivalent vertices

- The order of a unitrivalent tree is the number of trivalent vertices.
- The order of a Whitney disk W_J is the order of its (rooted) tree J.
- The order of an intersection point p is the order of its (un-rooted) tree t_p.

Order n Whitney towers in B^4

Definition

- A Whitney tower of order zero is a collection of properly immersed disks in B^4 bounding a framed link in S^3.
- For $n \geq 1$, a Whitney tower of order n is
  an order $n-1$ Whitney tower $\mathscr{W}$ together with
  Whitney disks pairing all order $n-1$ intersections in $\mathscr{W}$.

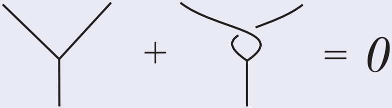


The target $\mathscr{T}_n$ for higher-order intersection invariants

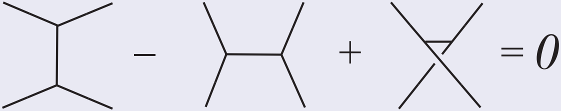
Definition

$\mathscr{T}_n$ is the abelian group generated by order n trees modulo the following antisymmetry and IHX (local) relations:

AS:



IHX:



The intersection invariant $\tau_n(\mathscr{W}) \in \mathscr{T}_n$

Definition

The order n intersection tree of an order n Whitney tower $\mathscr{W}$ is defined by

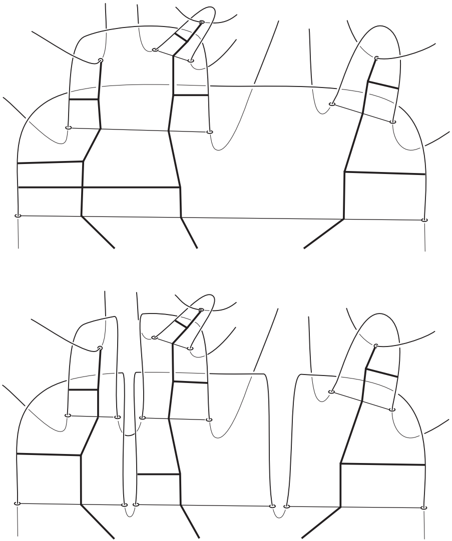
$$\tau_n(\mathscr{W}) := \sum \varepsilon_p \cdot t_p \in \mathscr{T}_n$$

The sum is over all order n intersections p,

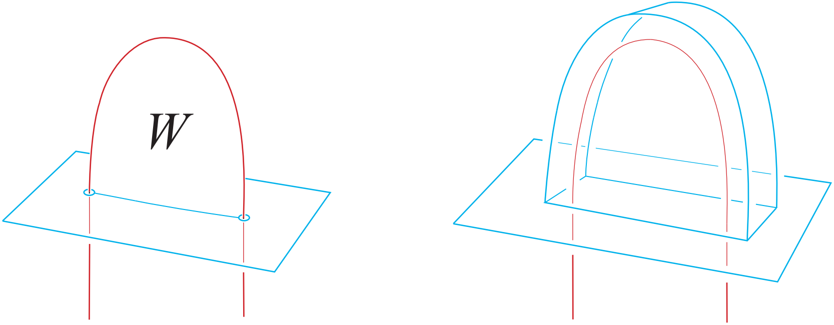
Theorem (order-raising)

If L bounds $\mathscr{W}$ with $\tau_n(\mathscr{W}) = 0 \in \mathscr{T}_n$,
then L bounds an order $n+1$ Whitney tower.

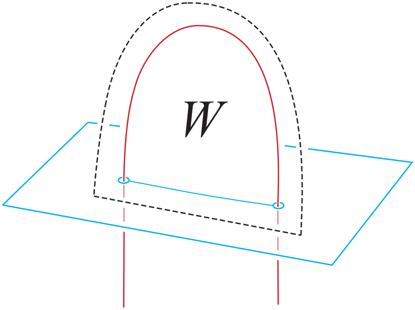
Splitting a Whitney tower.

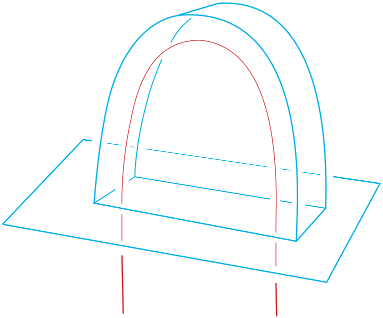


A successful Whitney move



depends on a framed Whitney disk

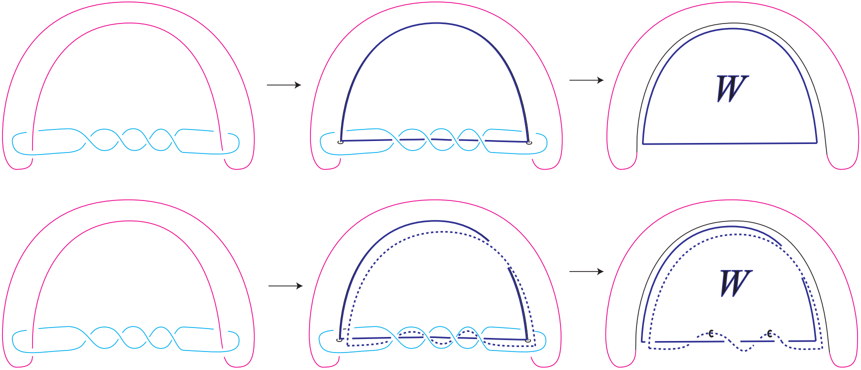




The Whitney section over ∂W extends over W.

Twisted Whitney disks

$W \mapsto \omega(W) \in \mathbb{Z}$ (relative Euler number) via orientation conventions.



A framed Whitney disk is 0-twisted.

Twisted trees

For any rooted tree J, define the ∞-tree by labeling the root with the symbol ∞:

$$J^{\infty} := \infty \text{---} J$$

∞-trees will be assigned to twisted Whitney disks, with the symbol ∞ representing a "twist".

Twisted Whitney towers in B^4

A twisted Whitney tower is allowed to have twisted Whitney disks.

It turns out that twisted order n Whitney disks "behave like" order $2n$ obstructions in the Whitney tower obstruction theory.

The target $\mathscr{T}_n^\infty$ for twisted intersection invariants

Definition

The abelian group $\mathscr{T}_{2n-1}^\infty$ is the quotient of $\mathscr{T}_{2n-1}$ by boundary-twist relations:

$$i \,-\!\!\!<^{J}_{J} = 0$$

Here J ranges over all order $n-1$ rooted trees .

The abelian group $\mathscr{T}_{2n}^\infty$ is gotten from $\mathscr{T}_{2n}$ by including order n ∞-trees as new generators and introducing (in addition to antisymmetry and IHX relations on non-∞ trees) new symmetry, twisted IHX, and interior-twist relations :

$$J^\infty = (-J)^\infty \qquad I^\infty = H^\infty + X^\infty - \langle H, X\rangle \qquad 2\cdot J^\infty = \langle J, J\rangle$$

$\mathscr{T}_{2n}^\infty$ is the universal quadratic refinement of the $\mathscr{T}_{2n}$-valued Whitney disk intersection form.

The twisted intersection invariant τ^{∞}

$$\tau_n^{\infty}(\mathscr{W}) := \sum \varepsilon_p \cdot t_p + \sum \omega(W_J) \cdot J^{\infty} \in \mathscr{T}_n^{\infty}$$

Theorem

If L bounds $\mathscr{W}$ with $\tau_n^{\infty}(\mathscr{W}) = 0 \in \mathscr{T}_n^{\infty}$, then L bounds an order $n+1$ twisted Whitney tower.

The twisted Whitney tower filtration

$\mathbb{W}_n^\infty := \{$framed links $L \subset S^3$ bounding order n twisted $\mathscr{W} \subset B^4\}$.

Get filtration: $\cdots \subseteq \mathbb{W}_3^\infty \subseteq \mathbb{W}_2^\infty \subseteq \mathbb{W}_1^\infty \subseteq \mathbb{W}_0^\infty$ (which factors through concordance).

$\mathsf{W}_n^\infty := \mathbb{W}_n^\infty$ modulo order $(n+1)$ twisted Whitney tower concordance.

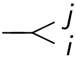
Proposition

W_n^∞ *is a finitely generated group under band sum.*

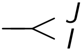
Will compute W_n^∞ by relating τ_n^∞ to Milnor and higher-order Arf invariants.

Rooted trees and the free Lie algebra $\mathscr{L}$

$\mathscr{L} = \oplus \mathscr{L}_n$ is the free $\mathbb{Z}$-Lie algebra on $\{X_1, \ldots, X_m\}$.

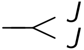


$$\longleftrightarrow \quad [X_i, X_j]$$



$$\longleftrightarrow \quad [I, J]$$

$\mathscr{L}_n$ is isomorphic to order $n-1$ *rooted* trees, modulo IHX and self-annihilation:



$$= \quad 0$$

From unrooted to rooted trees

The map $\eta_n^{\infty} : \mathscr{T}_n^{\infty} \to \mathscr{L}_1 \otimes \mathscr{L}_{n+1}$ sums over all choices of root on non-∞ trees:

$$\begin{aligned}
\eta_1^{\infty}\big(1 \,{-}\!\!\!< {}^{3}_{2}\big) \quad &= \quad X_1 \otimes {-}\!\!\!< {}^{3}_{2} \;+\; X_2 \otimes 1 \,{-}\!\!\!< {}^{3} \;+\; X_3 \otimes 1\,{-}\!\!\!< {}_{2} \\
&= \quad X_1 \otimes [X_2, X_3] + X_2 \otimes [X_3, X_1] + X_3 \otimes [X_1, X_2],
\end{aligned}$$

or

$$\begin{aligned}
\eta_2^{\infty}\big({}^{1}_{2} >\!\!{-}\!\!< {}^{2}_{1}\big) &= 2\big(X_1 \otimes {}_{2} >\!\!{-}\!\!< {}^{2}_{1} + X_2 \otimes {}^{1} >\!\!{-}\!\!< {}^{2}_{1}\big) \\
&= 2(X_1 \otimes [X_2, [X_1, X_2]] + X_2 \otimes [[X_1, X_2], X_1]),
\end{aligned}$$

and is defined on ∞-trees J of order $n/2$ by

$$\eta_n^{\infty}(J^{\infty}) := \frac{1}{2}\eta_n^{\infty}(\langle J, J \rangle).$$

Twisted Whitney towers and Milnor's μ_n invariants

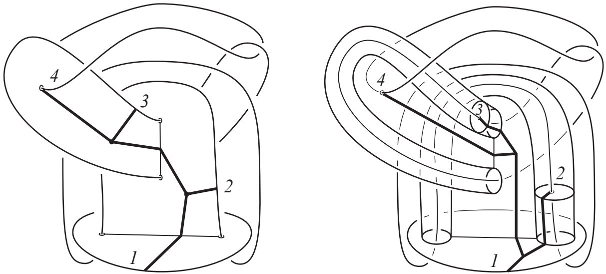
Both η_n^∞ and μ_n map onto $\mathsf{D}_n := \mathrm{Ker}(\mathscr{L}_1 \otimes \mathscr{L}_{n+1} \xrightarrow{[\ ,\ ]} \mathscr{L}_{n+2}) \cong \mathbb{Z}^{N_n}$.

Theorem

The first non-vanishing order n Milnor invariant defines a surjection $\mu_n : \mathsf{W}_n^\infty \to \mathsf{D}_n$, *and*

$$\mu_n(\partial \mathscr{W}) = \eta_n^\infty \circ \tau_n^\infty(\mathscr{W})$$

Proof uses (twisted) Whitney tower–grope correspondence, and (twisted) grope duality:



Computing W_n^∞.

Computation of W_n^∞ in "3/4" of the cases follows from the following theorem:

Theorem

$\eta_n^\infty : \mathscr{T}_n^\infty \to \mathsf{D}_n$ *are isomorphisms for* $n \equiv 0, 1, 3 \mod 4$.

Proof uses discrete Morse theory on chain complexes. (Inspired by J. Levine's extension of the Hall basis algorithm to free quasi-Lie algebras.)

Corollary

$W_n^\infty \cong \mathscr{T}_n^\infty \cong \mathsf{D}_n$ *for* $n \equiv 0, 1, 3 \mod 4$.

For the remaining cases $n = 4k - 2$, will define higher-order Arf invariants on the kernel of μ_{4k-2}...

Higher-order Arf invariants.

Theorem

There is an exact sequence (which is short exact for $k = 1$):

$$\mathbb{Z}_2 \otimes \mathsf{L}_k \xrightarrow{\alpha_k} \mathsf{W}^{\infty}_{4k-2} \xrightarrow{\mu_{4k-2}} \mathsf{D}_{4k-2} \to 0.$$

So for links in $\mathrm{Ker}(\mu_{4k-2})$ the only remaining obstructions to lying in $\mathsf{W}^{\infty}_{4k-1}$ are the following higher-order Arf invariants:

Definition

Define the higher-order Arf invariants Arf_k by

$$\mathrm{Arf}_k\colon \mathrm{Ker}(\mu_{4k-2}) \to (\mathbb{Z}_2 \otimes \mathsf{L}_k)/\mathrm{Ker}(\alpha_k)$$

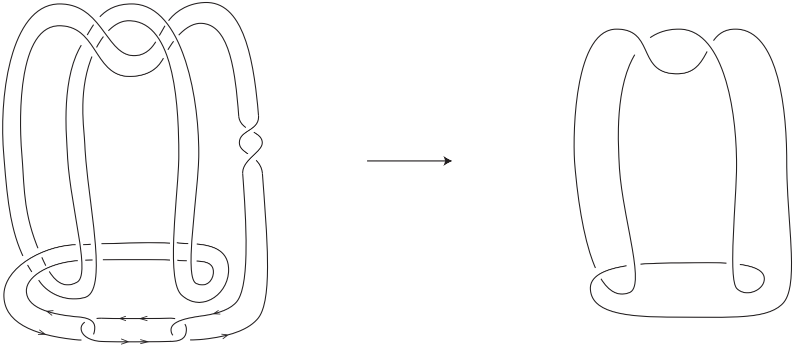
Corollary

The groups W^{∞}_n are classified by Milnor invariants μ_n and the above Arf invariants Arf_k for $n = 4k - 2$.

Arf_k

- The classical Arf invariants of the link components are $\mathrm{Arf}_1 \in \mathbb{Z}_2 \otimes \mathsf{L}_1 \cong \mathbb{Z}_2^m$.
- The map α_k takes $1 \otimes [J] \in \mathbb{Z}_2 \otimes \mathscr{L}_k$ to a link bounding $\mathscr{W}$ with $\tau^{\infty}_{4k-2}(\mathscr{W}) = (J, J)^{\infty}$.
- We conjecture that α_k is injective, so that Arf_k takes values in $\mathbb{Z}_2 \otimes \mathsf{L}_k$ for all k, and is determined by $\tau^{\infty}_{4k-2}(\mathscr{W}) \in \mathrm{span}\{(J, J)^{\infty}\} < \mathscr{T}^{\infty}_{4k-2}$.
- First open case is $k = 2$: For $L =$ Bing double of the figure-8 knot, $L \in \mathrm{Ker}\,\mu_6$, and L bounds $\mathscr{W}$ with $\tau^{\infty}_6(\mathscr{W}) = ((1,2),(1,2))^{\infty}$, so $\mathrm{Arf}_2(L) = [X_1, X_2]$ which generates $\mathbb{Z}_2 \otimes \mathsf{L}_2$.

The Bing double of the figure-8 knot

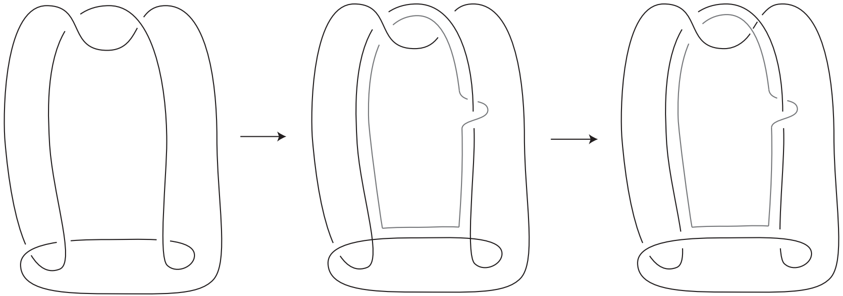


L = Bing(figure-8) bounds an order 6 twisted Whitney tower $\mathscr{W}$:

$$\mathscr{W} = D_1 \cup D_2 \cup W_{(1,2)} \cup W_{((1,2),(1,2))}.$$

$W_{(1,2)}$ is the trace of a null-homotopy of the figure-8 knot on the right.

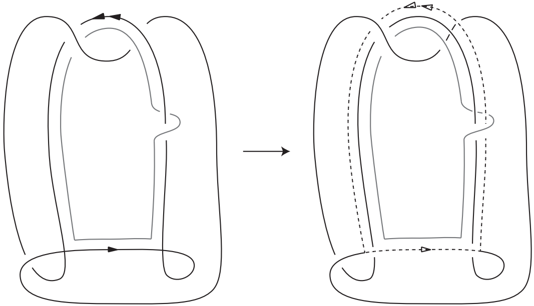
The Bing double of the figure-8 knot



The trace of the figure-8 knot null-homotopy describing $W_{(1,2)}$ has a canceling pair of self-intersections admitting an embedded twisted Whitney disk $W_{((1,2),(1,2))}$.
(The right-most picture is the unlink.)

The Bing double of the figure-8 knot

The second and third pictures from the previous frame:



The twisting $\omega(W_{((1,2),(1,2))}) = 1$ corresponds to the $+1$-linking between the 'inner' boundary component of a collar on $\partial W_{((1,2),(1,2))}$ and a Whitney section.
So $\tau_6^{\infty}(\mathscr{W}) = ((1,2),(1,2))^{\infty}$.

Appendix 1: The first non-vanishing Milnor invariant of a link L

- Consider the link group $\Gamma := \pi_1(S^3 \setminus L)$ and assume inductively that the longitudes $\ell_1, \ldots, \ell_m$ lie in Γ_{n+1}, the $(n+1)$-st term of the lower central series. If F is the free group on $X_1, \ldots, X_m$ then

$$\frac{\Gamma_{n+1}}{\Gamma_{n+2}} \cong \frac{F_{n+1}}{F_{n+2}} \cong \mathscr{L}_{n+1}$$

- The first non-vanishing order n Milnor invariant $\mu_n(L)$ of L is defined inductively by

$$\mu_n(L) := \sum_{i=1}^{m} [X_i] \otimes [\ell_i] \in \mathscr{L}_1 \otimes \mathscr{L}_{n+1}$$

- $\mu_n(L)$ actually lies in $\mathrm{D}_n := \mathrm{Ker}(\mathscr{L}_1 \otimes \mathscr{L}_{n+1} \xrightarrow{[\ ,\ ]} \mathscr{L}_{n+2})$