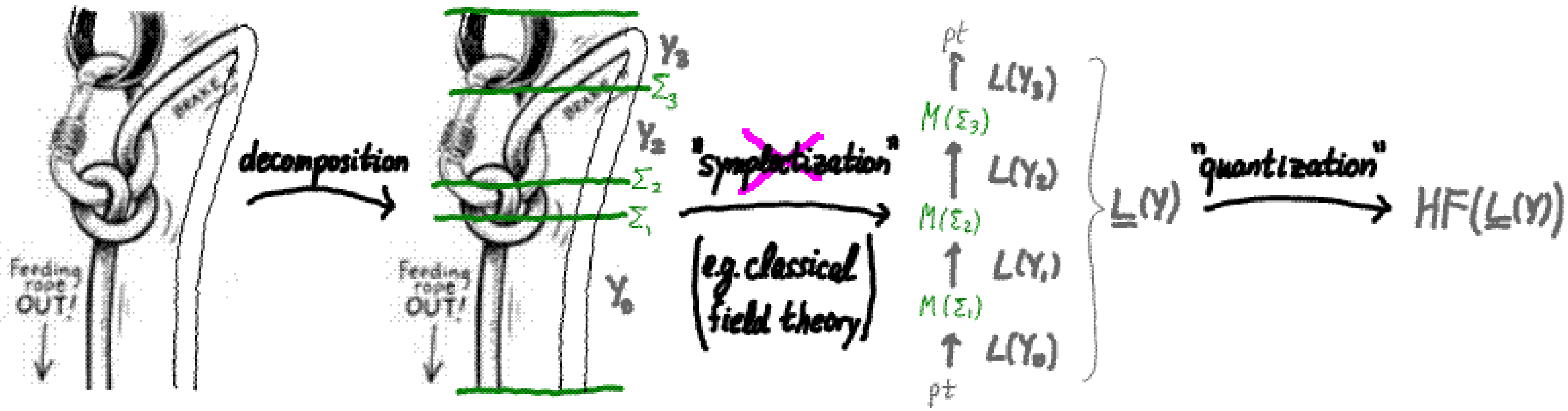
Low (3 and 4) dimensional manifolds and a symplectic 2-category



[Katrin Wehrheim – joint work with Chris Woodward
& (A_∞ chain level version) Sikimeti Ma'u
slides&preprints : www-math.mit.edu/~katrin]

1980s Weinstein : "Everything is Lagrangian" in symplectic topology

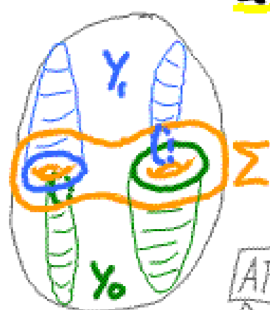
Gromov : pseudoholomorphic curves as tools —— " ——

1990s Atiyah-Floer; Ozsvath-Szabo : topological 3-manifold invariants via "magic":

(Heegård) splitting

$Y = Y_0 \cup_\Sigma Y_1$

3-manifold



& "representation" as pair of

L_{Y_0} Lagrangian

$\cap$

M_Σ symplectic

$\cup$

L_{Y_1} Lagrangian

Floer homology

$HF(L_{Y_0}, L_{Y_1})$

pseudoholomorphic curves

AF

$M_\Sigma = \{\tilde{\rho} : \pi_1(\Sigma) \to SU(2)$ group hom$\}/_{SU(2)}$

$\cup \qquad \downarrow$

$L_Y = \{\rho : \pi_1(Y) \to SU(2)$ group hom$\}/_{SU(2)}$

OS

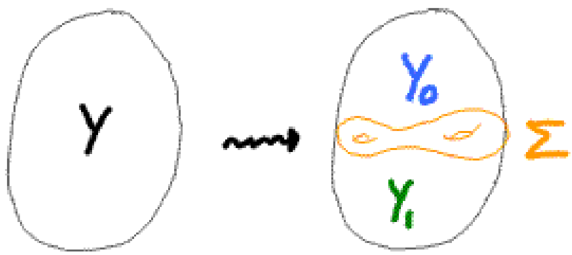
$M_\Sigma = Sym^g \Sigma = \{(z_1 \dots z_g) \in \Sigma^g\}/$permutation

$\cup$

$L_Y = \{(z_1 \dots z_g) \mid z_k \in$ k-th attaching circle$\}/$permutation "magic"

Atiyah-Floer conjecture : This defines a topological invariant,

Ozsvath-Szabo theorem : ie. is independent of the choice of splitting



$\rightsquigarrow$

L_{Y_0} Lagrangian
$\cap$
M_Σ symplectic
$\cup$
L_{Y_1} Lagrangian

$\rightsquigarrow$ Floer homology $HF(L_{Y_0}, L_{Y_1})$

[AF]: SU(2)-representation spaces
[OS]: $Sym^g(\Sigma)$ & attaching cycles

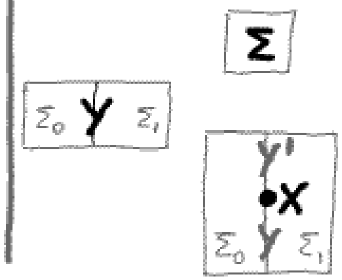
pseudoholomorphic curves / quilts

2010s
categorical interpretation:

$$(S^2 \xrightarrow{Y} S^2) = (S^2 \xrightarrow{Y_0} \Sigma \xrightarrow{Y_1} S^2) \overset{\text{functor}}{\rightsquigarrow} (\underset{\text{point}}{\underset{\shortparallel}{M_{S^2}}} \xrightarrow{L_{Y_0}} M_\Sigma \xrightarrow{L_{Y_1}} \underset{\text{point}}{\underset{\shortparallel}{M_{S^2}}}) \overset{\text{2-category structure}}{\rightsquigarrow} {}^2\mathrm{Mor}(L_{Y_0}, L_{Y_1}^T)$$

Examples of 2-categories & string diagram notation

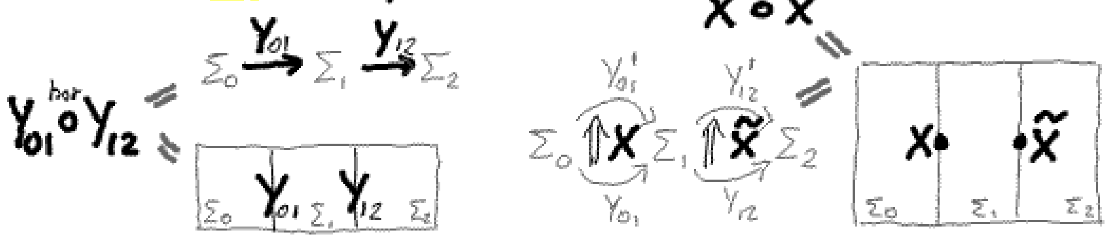
	Cat	notation	$\mathcal{T}op_{2+1+1}$	string notation
• object:	category	Σ	closed, oriented 2-manifold	
• morphism:	functor	$\Sigma_0 \xrightarrow{Y} \Sigma_1$	3dim. cobordism	
• 2-morphism:	natural transformation	$\Sigma_0 \Uparrow X \; \Sigma_1$	4dim. cobordism of cobordism	



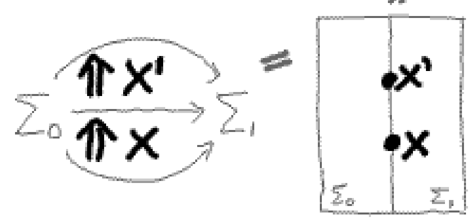
• horizontal composition

$Y_{01} \overset{hor}{\circ} Y_{12} = \Sigma_0 \xrightarrow{Y_{01}} \Sigma_1 \xrightarrow{Y_{12}} \Sigma_2$

$X \overset{hor}{\circ} \tilde{X}$

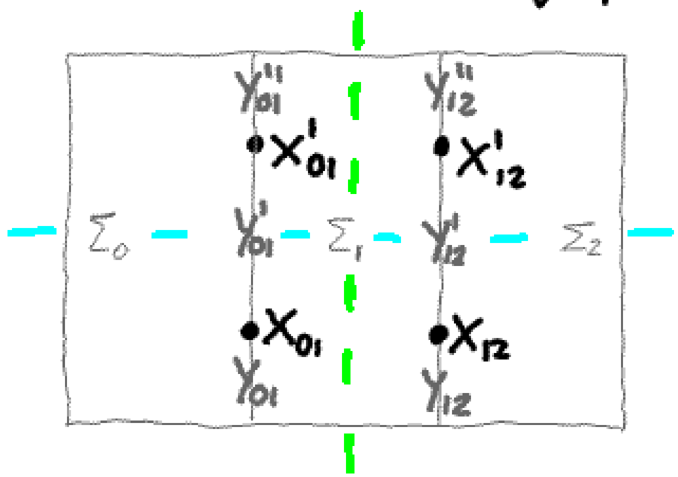


• vertical composition $X \overset{ver}{\circ} X'$



• 2-category axioms : larger string diagrams make sense

Main 2-category axiom:



defines a 2-morphism

$$
\begin{tikzcd}
\Sigma_0 \arrow[rr, bend left, "Y''_{01} \overset{hor}{\circ} Y''_{12}"] \arrow[rr, bend right, "Y_{01} \overset{hor}{\circ} Y_{12}"'] \arrow[rr, phantom, "\Uparrow X"] & & \Sigma_2
\end{tikzcd}
$$

that is

$$X = \underline{(X_{01} \overset{ver}{\circ} X'_{01}) \overset{hor}{\circ} (X_{12} \overset{ver}{\circ} X'_{12})} = \underline{(X_{01} \overset{hor}{\circ} X_{12}) \overset{ver}{\circ} (X'_{01} \overset{hor}{\circ} X'_{12})}$$

Def^{n} of symplectic category Symp$^{\#}$ (homology version)

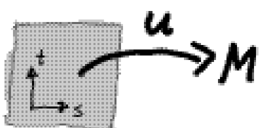
via realization of string diagrams as quilts

• **objects**: (M,ω) symplectic manifold

IS locally $(\mathbb{C}^n, \underbrace{\sum dz_j \wedge d\bar{z}_j}_{\text{preserved by transition maps}})$ $\rightsquigarrow$ carry compatible almost complex structures $J: M \to \mathrm{End}(TM)$, $J^2 = -\mathbb{1}$ (but $\nabla J \neq 0$)

$\omega(\cdot, J\cdot)$ metric

$\boxed{M}$ represents a well posed (elliptic, Fredholm) nonlinear PDE:



"patch"

$$\bar{\partial}_J u := \partial_s u + J(u)\partial_t u = 0$$

"pseudoholomorphic curve"

[Gromov, ...]

- **elementary morphisms** : $M \xrightarrow{L} N$ Lagrangian correspondence

$L \subset (M \times N, \underbrace{(-\omega_M) \times \omega_N}_{})$ submanifold, $TL \oplus J \cdot TL = T(M \times N)$

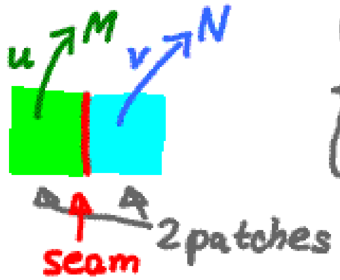
$(-J_M) \times J_N =: J$ (locally $L \subset M \times N \simeq \mathbb{R}^{m+n} \subset \mathbb{C}^{m+n}$)

Examples:

- **split** : $M_0 \xrightarrow{L_0 \times L_1} M_1$; $L_i \subset M_i$ Lagrangian ; also $M_0 \xrightarrow{L_0} pt$, $pt \xrightarrow{L_1} M_1$
- **graph**: $M \xrightarrow{gr\,\varphi} N$; $\varphi : (M, \omega_M) \to (N, \omega_N)$ symplectomorphism : $\varphi^* \omega_N = \omega_M$
- **symplectic quotient**: $M \xrightarrow{(\iota \times pr)\mu^{-1}(0)} M/\!\!/G = \mu^{-1}(0)/G$; $G \to Symp(M, \omega_M)$, $\mu : M \to Lie\, G$ moment map

eg. $\mathbb{C}^{n+1} \longrightarrow \mathbb{CP}^n = \{ |z_0|^2 + \ldots + |z_n|^2 - 1 = 0 \} / S^1$

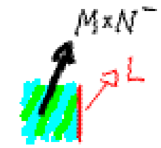
$M \overset{L}{|} N$ represents a well posed (elliptic, Fredholm) nonlinear PDE:



$$\begin{cases} \bar\partial_{J_M} u = 0 \\ \bar\partial_{J_N} v = 0 \end{cases}, \quad u \times v(\text{seam}) \subset L$$

"pseudoholomorphic quilt"

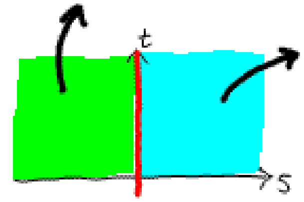
[Perutz, W-W]

(locally )

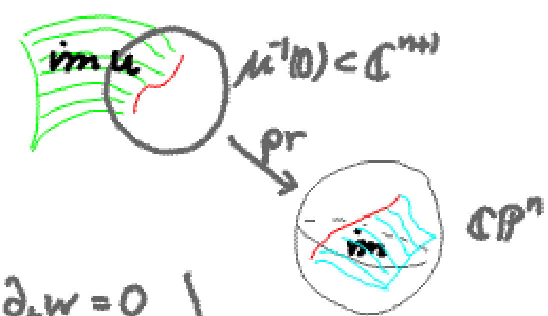
Example:

$\mathbb{C}^{n+1} \overset{\mu^{-1}(0)}{|} \mathbb{CP}^n$

$u(s,t) \in \mathbb{C}^{n+1} \quad \forall s<0$

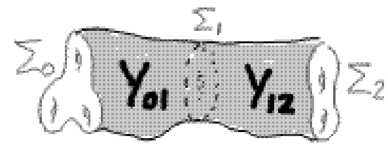


$v(s,t) \in \mathbb{CP}^n \quad \forall s>0$



$$\left(w(s,t) := (u(s,t), v(-s,t)) \ \text{satisfies} \ \begin{cases} \partial_s w + (i \times (-i)) \partial_t w = 0 \\ w(0,t) \in (1 \times pr)\,\mu^{-1}(0) \end{cases} \right)$$

In [Ozsvath-Szabo] [Atiyah-Floer] examples, topological composition corresponds to geometric composition of Lagrangian correspondences:



$$M(\Sigma_j) = \{ \tilde{\rho} : \pi_1(\Sigma_j) \to SU(n) \} / SU(n) \cong \frac{\text{flat } SU(n)\text{-connections}}{\text{gauge}}$$

$$M(\Sigma_0) \times M(\Sigma_1) \supset L(Y_{01}) \cong \left\{ \begin{array}{l} \text{flat } SU(n)\text{-connections} \\ \text{on } Y_{01} \text{ mod gauge} \end{array} \right\} \Big|_{\partial Y_{01} = \Sigma_0 \cup \Sigma_1}$$

$$\Rightarrow L(Y_{01} \cup Y_{12}) = L(Y_{01}) \circ L(Y_{12})$$

$$\begin{tikzcd} M_0 \arrow[r, "L_{01}"] \arrow[rr, bend right, "L_{01} \circ L_{12}"'] & M_1 \arrow[r, "L_{12}"] & M_2 \end{tikzcd}$$

$$\boxed{L_{01} \circ L_{12} := \pi_{M_0 \times M_2} \left(L_{01} \times L_{12} \cap M_0 \times \Delta_{M_1} \times M_2 \right)}$$

is an immersed Lagrangian if $\pitchfork$

call it "embedded" if $\pitchfork$ and π injective

• **morphisms** $Mor(M,N)$: $M = M_0 \xrightarrow{L_{01}} M_1 \to \dots \xrightarrow{L_{(k-1)k}} M_k = N$ (sequence of elementary morphisms)

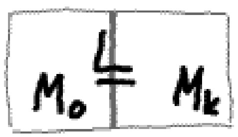
generalized Lagrangian correspondences

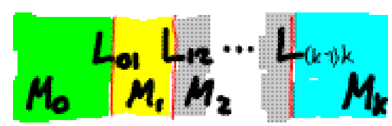
→ (horizontal) 1 **composition** of $M_0 \to \overset{L_{01}}{\dots} \to M_1$ and $M_1 \to \overset{L_{12}}{\dots} \to M_2$

$\boxed{L_{01} \# L_{12}} := M_0 \to \overset{L_{01}}{\dots} \to M_1 \to \overset{L_{12}}{\dots} \to M_2$ always defined, associative,

unit $1_{M_1} = (M_1)$ empty sequence

(• could quotient by "$L_{01} \# L_{12} \sim L_{01} \circ L_{12}$ if embedded", but rather prove $\sim$ in 2-category)

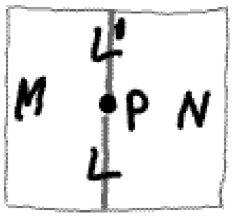
→ [

represents a PDE 

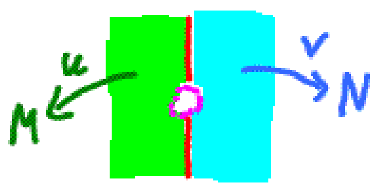
($k+1$ patches, k seams)]

• **2-morphisms**: Floer homology classes $\approx$ formal sums of intersection points

$$Mor(L,L') = HF(L,L') = H_*(CF,\partial) \qquad \partial \circlearrowleft CF := \sum_{p\in L\cap L'} \mathbb{Z}\langle p\rangle \quad \text{if } L \pitchfork L'$$



represents a well posed (elliptic, Fredholm) nonlinear PDE:



$$\left\{\begin{array}{ll} \bar\partial_{J_M} u = 0 & u\times v\left(\begin{smallmatrix}\text{upper}\\ \text{seam}\end{smallmatrix}\right) \subset L' \\ \bar\partial_{J_N} v = 0 & u\times v\left(\begin{smallmatrix}\text{lower}\\ \text{seam}\end{smallmatrix}\right) \subset L \\ (u\times v)(\text{puncture}) \longrightarrow p & \end{array}\right\}$$

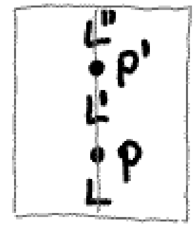
"pseudoholom. quilt" [W-W]

similarly for generalized correspondences: quilted Floer homology generated by

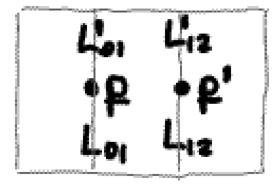
$$\cap\left(\begin{tikzcd} M \arrow[r, bend left, "L"] \arrow[r, bend right, "L'"'] & N \end{tikzcd}\right) = \cap\left(\begin{tikzcd} M_0 \arrow[r, "L_{01}"] & M_1 \arrow[r, "L_{12}"] & \cdots \\ & M_{d-1} \arrow[ul, "L_{(d-1)d}"] & \cdots \end{tikzcd}\right) := \left\{ \underline{p} = (p_0,\dots,p_{d-1}) \;\middle|\; (p_{i-1},p_i)\in L_{(i-1)i} \right\}$$

* string diagram defines 2-morphism by viewing boundary as puncture
  ⇝ vertical and horizontal ^{2}composition
  and ^{2}identity $1_L := \boxed{L}$

$$\sum_{q \in \cap \text{ Lagrangians on seams to outside}} \#\left\{ \begin{array}{l} \text{pseudoholom. quilts} \\ \text{w. outside puncture} \to q \end{array} \right\} \langle q \rangle$$

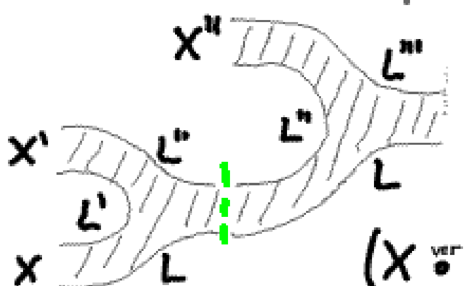


and



* gluing laws for pseudoholomorphic quilts ⇒ **this defines a 2-category**
  (when restricting to monotone/exact objects & morphisms)

Ex: $\overset{ver}{\circ}$ associativity for M=pt



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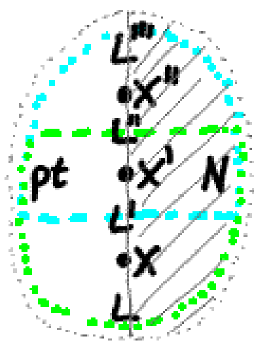

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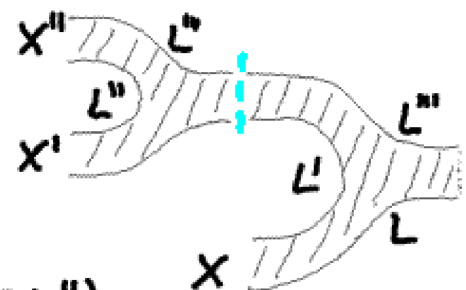


$(X \overset{ver}{\circ} X') \overset{ver}{\circ} X''$ $\qquad X \overset{ver}{\circ} (X' \overset{ver}{\circ} X'')$

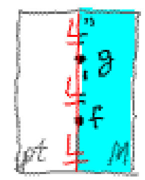
Corollary: There is a 2-functor $\mathcal{D}on : Symp^{\#} \longrightarrow \mathcal{C}at = \begin{pmatrix} \text{categories} \\ \text{functors} \\ \text{nat. transf}^{\text{ns}} \end{pmatrix}$

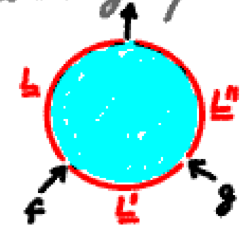
induced by the 2-category structure and choice of base object $M_{base} = pt$.

- M symplectic $\longmapsto \mathcal{M}or(pt, M)$ "homology of extended Fukaya category"

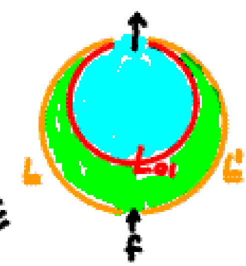
  objects $pt \to \underline{L} \to M$

  $\mathcal{M}or(\underline{L}, \underline{L}') = HF(\underline{L}, \underline{L}')$

  $f \circ g =$ 

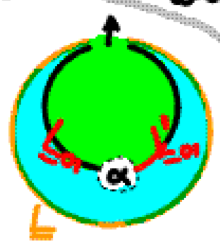
  $\hat{=}$ 

- $M_0 \xrightarrow{L_{01}} M_1$ Lagr. corr. $\longmapsto$ functor $\mathcal{D}on(L_{01}) \begin{cases} \mathcal{M}or(pt, M_0) \longrightarrow \mathcal{M}or(pt, M_1) \\ pt \to \underline{L} \to M_0 \longmapsto pt \to \underline{L} \to M_0 \xrightarrow{L_{01}} M_1 \\ HF(\underline{L}, \underline{L}') \to HF(\underline{L} \# L_{01}, \underline{L}' \# L_{01}) \end{cases}$

$\cong$ 

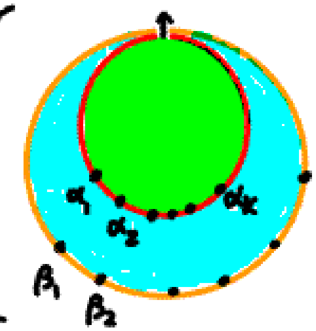
- $HF(\underline{L}_{01}, \underline{L}_{01}') \ni \alpha$

  $\downarrow$

  natural transformation 

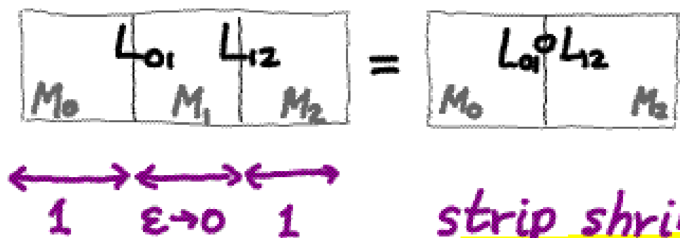
$= id_{\underline{L}} \# \alpha$

The chain-level / A_∞-functor is constructed by counting pseudoholomorphic quilts in families of quilted surfaces realizing associahedra / multiplihedra / ... polytopes.

$Q^{mult}_{k,r} = \Big\{$  $\Big\}$

Thm: $M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2$ is isomorphic to $L_{01} \circ L_{12}$ if $L_{01} \times L_{12} \pitchfork M_0 \times \Delta_{M_1} \times M_2 \xrightarrow{\pi_{M_0 \times M_2}} L_{01} \circ L_{12}$ "embedded"

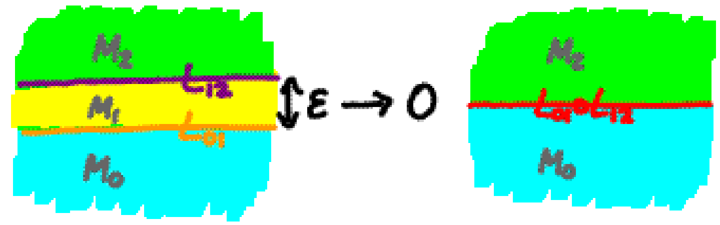
and "bubbling is excluded a priori".

Proof: 

for counting pseudoholomorphic quilts

by strip shrinking

possible bubbling in strip shrinking



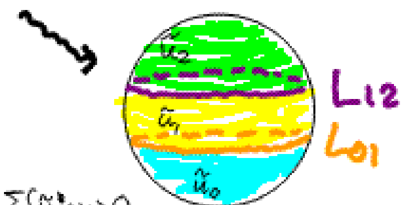
$S^2 \to M_0$ $\quad$ $S^2 \to M_1$ $\quad$ $S^2 \to M_2$

$D^2 \to M_0^- \times M_1$ $\quad$ $D^2 \to M_1^- \times M_2$

"figure eight"

removal of singularity unclear

but nearby quilt "bubbles off some energy"



$0 < \sum_{i=0}^{2} \int |du_i|^2 = \sum \int u_i^*\omega_i \wr \sum \int \tilde{u}_i^*\omega_i > 0$

$D^2 \to M_0^- \times M_2$

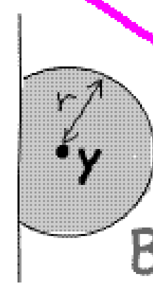
bubble exclusion: by control of energy (e.g. monotone symplectic and Lagrangian manifolds)

and **mean value inequalities**, for $e : \mathbb{H}^n \to [0,\infty)$

$\mathbb{H}^n = \{ z \in \mathbb{R}^n \mid z_1 \geq 0 \}$

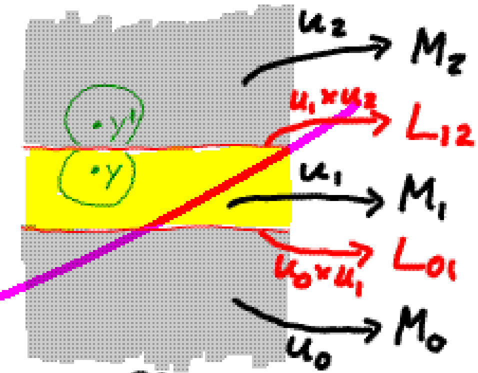
$\forall n \geq 2 \;\exists C \;;\; \forall a,b \geq 0 \;\exists \hbar(a,b) > 0 \;:\; \forall y \in \mathbb{H}^n,\, r > 0$

$$\left.\begin{aligned} -\sum_{i=1}^{n} \partial_i^2 e =: \Delta e &\leq A_0 + a\, e^{\frac{n+2}{n}} \\ -\partial_1 e =: \left.\frac{\partial}{\partial \nu}\right|_{\partial \mathbb{H}^n} e &\leq B_0 + b\, e^{\frac{n+1}{n}} \\ \int_{B_r(y) \cap \mathbb{H}^n} e &\leq \hbar(a,b) \end{aligned}\right\} \Rightarrow e(y) \leq C \left(r^{-n} \int_{B_r(y) \cap \mathbb{H}^n} e + A_0 r^2 + B_0 r \right)$$



$B_r(y) \cap \mathbb{H}^n$

Sketch of bubble exclusion: $n=2$ $\quad e_i = |du_i|^2$



- $\bar{\partial}_{J_i} u_i = 0 \quad \Rightarrow \quad \Delta e_i \le a \cdot e_i^2$
- $(u_i \times u_{i+1})(\text{seam}) \in L_{i(i+1)} \quad \Rightarrow \quad \frac{\partial}{\partial r}(e_i + e_{i+1})\Big|_{\partial H^2} \le b\,(e_i+e_{i+1})^{3/2}$

$\Rightarrow$ If singularity forms (for a sequence $(u_0,u_1,u_2)_{k\in\mathbb{N}}$) then energy concentrates

$$e_1(y_k) + e_2(y_k') \xrightarrow[k\to\infty]{} \infty \qquad\qquad \int_{B_{r_k}(y,y')} e_1 + e_2 \ge \hbar(a,b) > 0 \qquad r_k \to 0$$

Thm: (monotone or exact) Symplectic manifolds and Lagrangian correspondences with geometric composition (when embedded) can be extended to a symplectic 2-category *Symp*.

Corollary: There is a 2-functor $Symp \longrightarrow Cat/_\sim = \begin{pmatrix} \text{categories} \\ \text{functors/isom.} \\ \text{nat. transf}^{\underline{n}}\text{/isom} \end{pmatrix}$

where *Symp* is $\begin{pmatrix} \text{admissible symplectic manifolds} \\ \text{adm. generalized Lagrangian corresp.}/\sim \text{ by embedded geometric composition} \\ \text{HF-classes} \end{pmatrix}$.

In particular

- $HF(\ldots L_{01}, L_{12} \ldots) \simeq HF(\ldots L_{01} \circ L_{12} \ldots)$ if $L_{01} \times L_{12} \pitchfork M_0 \times \Delta_1 \times M_2 \hookrightarrow L_{01} \circ L_{12}$
- $\underbrace{Don(L_{01}) \circ Don(L_{12}) = Don(L_{01} \# L_{12})}_{\text{always by functoriality}} \underbrace{\cong Don(L_{01} \circ L_{12})}_{\text{by } L_{01} \# L_{12} \simeq L_{01} \circ L_{12} \text{ if embedded}}$

Cor.: Any (partial)* functor $\mathcal{T}on_{n+1} \longrightarrow \mathcal{S}ymp$ gives rise to a "topological quantum field theory" $\mathcal{T}on_{n+1} \longrightarrow \mathcal{C}at$.

$*$

all objects: n-manifolds  ... Σ_g $\longrightarrow$ M_{Σ_g} symplectic

elementary morphisms: handle attachment / trivial cobordism $\longrightarrow$ $L_Y \subset M_{\Sigma_g}^- \times M_{\Sigma_{g+1}}$ Lagrangian

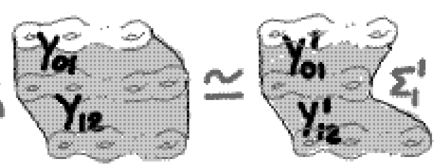
Σ_g  Y  $\Sigma\times[0,1]$ $\qquad L_{\Sigma\times[0,1]}$ = diagonal $\subset M_\Sigma^- \times M_\Sigma$

Σ_{g+1}

"moves" between prime decompositions: $Y_{01} \underset{\Sigma_1}{\cup} Y_{12} \simeq Y'_{01} \underset{\Sigma'_1}{\cup} Y'_{12}$ $\longrightarrow$ $L_{Y_{01}} \circ L_{Y_{12}} = L_{Y'_{01}} \circ L_{Y'_{12}}$

embedded geometric composition

- handle cancellation
- ~ switch
- (• ~ slide)
- diffeomorphism
- trivial cancellation $Y \cup (\Sigma\times[0,1]) \simeq Y$

Σ_1 

Σ'_1

[Cerf theory] $\longleftrightarrow$ [WW construction of Symp]

Cor.: Any (partial) functor $\mathcal{T}\!op_{n+1+1} \to \mathcal{S}ymp$ gives rise to a "topological quantum field theory" $\mathcal{T}\!op_{n+1+1} \to \mathcal{C}at$.

Ex.: Perutz-Lekili : 2+1 TQFT containing Ozsvath-Szabo "Heegård Floer" 3-mfd invariant

Perutz + work : 2+1+1 —"— conj. Seiberg-Witten 4-mfd —"—

W.-W. : 2+1 —"— Atiyah-Floer type 3-mfd —"—

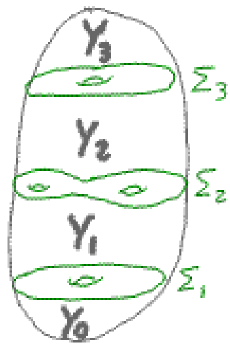
W.-W.-work : 2+1+1 —"— conj Donaldson type 4-mfd —"—

...

2010 3-dim topological invariants via "functoriality"

Y 3-manifold $\xrightarrow[\text{decompose into simple pieces}]{[\text{Cerf}, \ldots]}$ $Y = \bigcup_{i=1} Y_i$ $\xrightarrow{\text{go symplectic}}$ $\underline{L}(Y, \text{choice})$ $\xrightarrow[\text{count holomorphic curves}]{[WW]}$ $HF(\underline{L})$

! choice !



...

Bottaching cycles

symmetric products

SU(n) representations

pt

$\uparrow$ $L(Y_3)$

$M(\Sigma_3)$

$\uparrow$ $L(Y_2)$

$M(\Sigma_2)$

$\uparrow$ $L(Y_1)$

$M(\Sigma_1)$

$\uparrow$ $L(Y_0)$

pt

$HF(\underline{L}) \overset{?}{=} \mathrm{Mor}(\underline{L}, 1)$

$\downarrow$

topological invariant of Y

! independent of choice !

another sample invariant for $\left(\begin{array}{ll} Y & \text{closed 3-manifold} \\ [f: Y \to S^1] & \text{homotopy class of } S^1\text{-valued function} \end{array} \right)$

using

- cyclic connected Cerf theory [Gay-Kirby + W-W]: Decompositions of Y along regular, connected level sets of $f: Y \to S^1$ exist and are unique up to Cerf moves: cancellation / switch of critical points

- $\{\rho : \pi_1(\Sigma \setminus pt) \to SU(n) \text{ hom.} \mid \rho(\text{pt}) = -\mathbb{1}\} / SU(n)$ smooth, monotone symplectic

  $\{$ -"- $Y \setminus \text{line}$ —"— line -"- $\} / SU(n)$ —"— Lagrangian

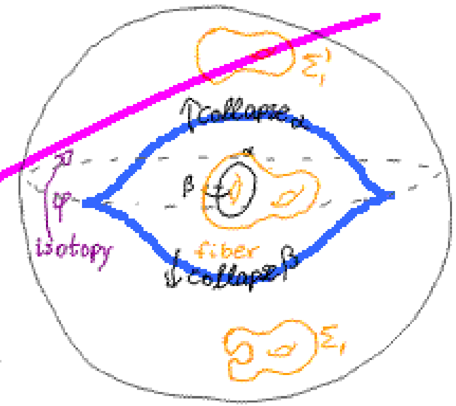
  compression body: all crit pts of same index (1 or 2)

next:

[Perutz, Lekili, Gay-Kirby, Williams]: $X^4 \xrightarrow{f} S^2$ broken Lefschetz fibration

exist (with connected fibers) and are unique up to moves

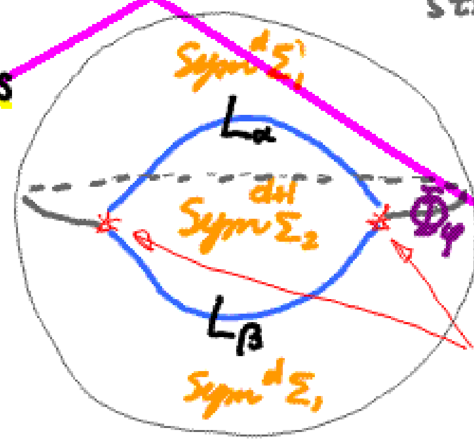
... and very close to a 2-categorical string diagram on S^2...



$\rightsquigarrow$ define 4-mfd invariants

by

- symplectic labeling
- checking moves



induced symplectomorphism

$1_\varphi \in HF(\Phi_\eta, \Phi_\varphi) \cong HF(L_\beta, L_\alpha, gr\Phi_\varphi)$