Midterm 2 Solutions, Math 1A, section 1

1. Find an equation of the tangent line to the curve at the given point.

$$y = \cos\big(-e^{-x} + 1\big) - 2, \quad (0, -1).$$

Solution: We compute $y' = \frac{dy}{dx}(\cos(-e^{-x}+1) - 2) = -\sin(-e^{-x}+1)\cdot(e^{-x})$ by two applications of the chain rule and rules for differentiating trig. and exponential functions. We plug in $y'(0) = -\sin(-e^{-0}+1)\cdot e^{-0} = 0$. Thus, the tangent line is given by $y = -1$.

2. Find $\frac{dy}{dx}$ by implicit differentiation

$$(x^2 + y)^3 = 3(x^2 - y).$$

Solution: We differentiate each side of the equation:

$$\frac{d}{dx}(x^2+y)^3 = \frac{d}{dx}3(x^2-y)$$
$$3(x^2+y)^2\cdot(2x+y') = 3(2x-y')$$

using the chain rule and power rule. Divide both sides by 3 and solve for y':

$$((x^2+y)^2+1)y' = 2x(1-(x^2+y)^2)$$

$$\frac{dy}{dx} = y' = 2x(1-(x^2+y)^2)/((x^2+y)^2+1)$$

where defined.

3. If $V(t)$ is the volume of a balloon inflated to radius $r(t)$ at time t, find dV/dt when the radius is 2 and $dr/dt = 7$.

Solution: We have the equation $V(t) = \frac{4}{3}\pi r(t)^3$, the formula for the volume of a ball in terms of the radius.

Differentiate both sides with respect to t:

$$\frac{d}{dt}V(t) = \frac{d}{dt}(\frac{4}{3}\pi r^3) = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

by the chain rule. We plug into this equation $r = 2$ and $\frac{dr}{dt} = 7$, to get

$$\frac{d}{dt}V(t) = 4\pi 2^2\cdot 7 = 112\pi.$$

4. Find the linearization $L(x)$ of the function $f(x) = \sinh(1-x)$ at $a = 1$.

Solution: Observe that $f(x)$ is defined and differentiable for all x, so it makes sense to talk about a linearization at $a = 1$. We know that $f'(x) = \cosh(1-x)\cdot(-1) = -\cosh(1-x)$ due to the chain rule, so $f'(1) = -\cosh(0) = -1$. Thus,

$$L(x) = f(1) + f'(1)(x-1) = 0 + (-1)(x-1) = 1 - x.$$

5. Find the absolute maximum and absolute minimum of the function $f(x) = 3x^4 - 4x^3 - 2$ on the interval $[-2, 2]$.

   Solution: We compute $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$, which exists for all x. Thus, the critical points are solutions to $f'(x) = 0 = 12x^2(x-1)$. The solutions are $x = 0, x = 1$, which are both critical points in the interval $[-2, 2]$. So we compute $f(-2) = 3\cdot(-2)^4 - 4\cdot(-2)^3 - 2 = 48 + 32 - 2 = 78$, $f(2) = 48 - 32 - 2 = 14$, $f(0) = -2$, $f(1) = 3 - 4 - 2 = -3$. So the maximum is realized by $f(-2) = 78$, and the minimum is realized by $f(1) = -3$.

6. Find the limit.

$$\lim_{x\to 0} \frac{e^{2x} - 1 - 2x}{x^2}.$$

   Solution: We have $\lim\limits_{x\to 0} e^{2x} - 1 - 2x = 0$ and $\lim\limits_{x\to 0} x^2 = 0$, so the limit is indeterminate of type $\frac{0}{0}$. We attempt to apply l'Hospital's theorem by taking the ratio of the derivative of the numerator and denominator.

$$\lim_{x\to 0} \frac{(e^{2x} - 1 - 2x)'}{(x^2)'} = \lim_{x\to 0} \frac{2e^{2x} - 2}{2x} = \lim_{x\to 0} \frac{e^{2x} - 1}{x}$$

   by the chain rule. This limit is still indeterminate of type $\frac{0}{0}$, so we take the ratio of the derivatives again:

$$\lim_{x\to 0} \frac{(e^{2x} - 1)'}{x'} = \lim_{x\to 0} \frac{2e^{2x}}{1} = 2e^{2\cdot 0} = 2.$$

   Thus, by two applications of l'Hospital's rule, we conclude

$$\lim_{x\to 0} \frac{e^{2x} - 1 - 2x}{x^2} = \lim_{x\to 0} \frac{(e^{2x} - 1)'}{x'} = \lim_{x\to 0} \frac{(e^{2x} - 1 - 2x)'}{(x^2)'} = 2$$

7. Sketch the curve $y = \frac{\ln x}{x^2}$. Your sketch should show the domain of the function, local maxima and minima, where the function is increasing or decreasing, any zeros of the function, the behavior for large values of x, and the behavior near $x = 0$ to determine any asymptotes, and determine any points where the function is not differentiable. You need not show concavity or points of inflection.

   Solution: Domain: The domain of the function is $x > 0$, since this is the domain of $\ln(x)$. The zero of the function is when $\ln(x) = 0$, so $x = 1$.

   We compute $y' = 1/x^3 - 2\ln(x)/x^3 = x^{-3}(1 - 2\ln(x))$. This is defined for all $x > 0$, so the only critical points are when $y' = 0$, which is for $1 - 2\ln(x) = 0$. Then $\ln(x) = 1/2$, so $x = e^{\ln(x)} = e^{1/2}$. This is the only critical point. The derivative is positive for $0 < x < e^{1/2}$, since then $1 - 2\ln(x) > 0$, and therefore $(1 - 2\ln(x))/x^3 > 0$. Similarly, $y' < 0$ for $e^{1/2} < x$, since $1 - 2\ln(x) < 0$, so $(1 - 2\ln(x))/x^3 < 0$.

   Thus, y is increasing for $0 < x < e^{1/2}$, and y is decreasing for $e^{1/2} < x$, by the I/D test.

   By the first derivative test, $e^{1/2}$ is a maximum of y.

   For $\lim\limits_{x\to 0^+} \frac{\ln(x)}{x^2} = -\infty/0^+ = -\infty$. So there is a vertical asymptote at $x = 0$.

   We have $\lim\limits_{x\to\infty} \frac{\ln(x)}{x^2}$ is indeterminate of type $\frac{\infty}{\infty}$. We compute $\lim\limits_{x\to\infty} \frac{\ln(x)'}{(x^2)'} = \lim\limits_{x\to\infty} \frac{1/x}{2x} = \lim\limits_{x\to\infty} \frac{1}{2x^2} = 1/\infty = 0$. By l'Hospital's rule, we get $\lim\limits_{x\to\infty} \frac{\ln(x)}{x^2} = \lim\limits_{x\to\infty} \frac{\ln(x)'}{(x^2)'} = 0$. So there is a horizontal asymptote for $y = 0$.

To plot, plug in $y(e^{1/2}) = \ln(e^{1/2})/(e^{1/2})^2 = 1/(2e)$ to compute the maximum value.