Midterm 2 Solutions, Math 1A, section 1

Thursday, November 6, 2008, 8:10-9:30 am

1. Let $p \neq 0$. Show by implicit differentiation that the tangent line to the curve

$$x^p + y^p = 1,\ x > 0,\ y > 0$$

at the point (x_0, y_0) is given by the equation $x_0^{p-1}x + y_0^{p-1}y = 1$. Show that the x-intercept a and y-intercept b of the tangent line satisfy $a^{p/(1-p)} + b^{p/(1-p)} = 1$ if $p \neq 1$.

Solution: Implicitly differentiate, using the power, sum, and chain rules:

$$\frac{d}{dx}(x^p + y^p) = px^{p-1} + py^{p-1}y' = \frac{d}{dx}(1) = 0.$$

Solving for y', we get $y' = -(x/y)^{p-1}$. We plug in the point (x_0, y_0) such that $x_0^p + y_0^p = 1$ into the point-slope formula for the tangent line:

$$y - y_0 = -(x_0/y_0)^{p-1}(x - x_0).$$

Multiplying by y_0^{p-1} and putting the constants to one side, we obtain

$$y_0^{p-1}y + x_0^{p-1}x = y_0^p + x_0^p = 1.$$

Setting x and $y = 0$, we see that the intercepts are $(a, b) = (x_0^{1-p}, y_0^{1-p})$. Then we see that

$$a^{p/(1-p)} + b^{p/(1-p)} = (x_0^{1-p})^{p/(1-p)} + (y_0^{1-p})^{p/(1-p)} = x_0^p + y_0^p = 1.$$

2. A ladder $10ft.$ long leans against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a speed of $2ft./s.$, how fast is the angle between the ladder and the wall changing when the bottom of the ladder is $6ft.$ from the base of the wall?

Solution: Let α be the angle between the ladder and the wall, let $x\ ft.$ be the distance from the base of the ladder and the wall, and let $t\ s.$ be the time. The given information implies that $\frac{dx}{dt} = 2$. Then $\sin(\alpha) = x/10$, so $\alpha = \arcsin(x/10)$. Then we differentiate using the chain and constant multiple rules:

$$\frac{d\alpha}{dt} = \frac{d}{dt}\arcsin(x/10) = \frac{1}{\sqrt{1-(x/10)^2}}\frac{dx}{dt}\frac{1}{10} = \frac{2}{\sqrt{10^2 - x^2}}.$$

When $x = 6$, we have

$$\frac{d\alpha}{dt}rad/s. = \frac{2}{\sqrt{10^2 - 6^2}}rad/s. = \frac{1}{4}rad/s.$$

One may also solve this problem using implicit differentiation.

3. Prove that $\ln(x) \leq x - 1$ for $x > 0$.

Solution: $\ln'(x) = 1/x = x^{-1}$ for $x > 0$. The tangent line to $\ln(x)$ at $x = 1$ is determined by $y - \ln(1) = \ln'(1)(x-1)$, so $y = x - 1$ since $\ln(1) = 0, \ln'(1) = 1$. We also compute that $\ln''(x) = \frac{d}{dx}x^{-1} = -x^{-2}$ by the power rule. For $x > 0$, $-x^{-2} < 0$, and thus by the concavity test, $\ln(x)$ is concave downwards for $x > 0$. This means that for all $x > 0$, $y = \ln(x)$ lies below every tangent line by the definition of concave downwards. In particular, $\ln(x) \leq x - 1$ for $x > 0$.

One may also use the increasing/decreasing test or the mean value theorem to solve this problem.

4. Let

$$g(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{1}$$

Show that g is differentiable and $g'(0) = 0$.

Solution: For $x > 0$, g is obtained as a composition of differentiable functions, and therefore is differentiable by the chain rule, power rule and constant multiple rule. For $x < 0$, g is constant, so is differentiable with derivative 0 by the constant rule. So we need only show that $g'(0) = 0$. We have

$$\lim_{h\to 0^-} \frac{g(h) - g(0)}{h} = \lim_{h\to 0^-} \frac{0 - 0}{h} = 0.$$

$$\lim_{h\to 0^+} \frac{g(h) - g(0)}{h} = \lim_{h\to 0^+} \frac{e^{-1/h} - 0}{h} = \lim_{h\to 0^+} \frac{1/h}{e^{1/h}}.$$

This second limit is indeterminate of the form ∞/∞, since $\lim\limits_{h\to 0^+} 1/h = \infty$. Then we compute $\frac{d}{dh} e^{1/h} = -1/h^2 e^{1/h}$ by the chain and power rules, and $\frac{d}{dh} 1/h = -1/h^2$ by the power rule. So we get

$$\lim_{h\to 0^+} \frac{-1/h^2}{-1/h^2 e^{1/h}} = \lim_{h\to 0^+} \frac{-1}{e^{1/h}} = -1/\infty = 0.$$

By l'Hospital's rule, we conclude that $\lim\limits_{h\to 0^+} \frac{g(h)-g(0)}{h} = 0$.

Thus, left and right limits agree, so we have $g'(0) = 0$ by the limit definition of the derivative, and therefore $g(x)$ is differentiable for all x.

5. Bismuth-210 has a half-life of 5.0 days. A sample of Bismuth-210 has a mass of $128mg$.

   (a) Find a formula for the mass of the sample remaining after t days.
   (b) Find the mass remaining after 30 days.
   (c) When is the mass reduced to $1mg$?

   Solution:

   (a) The mass is given by $M(t) = M(0)e^{kt}$, since it obeys the law of natural decay. The half-life is 5 days, so $M(5) = M(0)/2$. We may solve for k by letting $M(5) = M(0)e^{k5} = M(0)/2$. Thus, $e^{5k} = 1/2$, so $5k = \ln(1/2)$, and $k = -\ln(2)/5$. We are also given that $M(0) = 128$, so we get $M(t) = 128e^{-\ln(2)t/5} = 128 * 2^{-t/5}$. So $M(t)mg = 128 * 2^{-t/5}mg$.
   (b) $M(30)mg = 128 * 2^{-30/5}mg = 128/2^6 mg = 2mg$.
   (c) $M(35)mg = 128 * 2^{-35/5}mg = 1mg$. Thus the mass is reduced to $1mg$ in 35 days.

6. Find the maxima and minima of $y = x^3 - 3x + 1$ on the interval $[0, 3]$.

   Solution: The function is differentiable and therefore continuous on the entire interval. We compute the critical points of $x^3 - 3x + 1$ by computing the derivative (using sum, power, and constant multiple rules) and setting it equal to zero.

$$\frac{d}{dx}(x^3 - 3x + 1) = 3x^2 - 3 = 3(x - 1)(x + 1) = 0.$$

   The solutions of this equation are ± 1. Then $y(1) = -1$, $y(-1) = 3$ (but we do not need to include this value since it is not in the interval $[0, 3]$). Plugging in the endpoints, $y(0) = 1$, $y(3) = 19$. So the maximum is 19, and the minimum is -1 by the closed interval test.

7. Find the intervals on which f is increasing and decreasing, find the intervals of concavity and the inflection points, for the function $f(x) = (x^2 + 4x + 5)e^{-x}$.

Solution: Using the product rule, power rule, and chain rule, we have

$$f'(x) = (x^2 + 4x + 5)(-e^{-x}) + (2x + 4)e^{-x} = -(x^2 + 2x + 1)e^{-x} = -(x + 1)^2 e^{-x}.$$

Then we have $f'(x) \leq 0$, with equality only if $x = -1$. Thus, $f(x)$ is decreasing for all x by the decreasing test, since it is decreasing for both $x < -1$ and $x > -1$.

To determine the concavity, we compute $f''(x) = -2(x+1)e^{-x} - (x+1)^2(-e^{-x}) = (x^2-1)e^{-x}$ using the chain, product, and power rules. Then since $e^{-x} > 0$ for all x, $f''(x) > 0$ for $x^2 - 1 > 0$, and $f''(x) < 0$ for $x^2 - 1 < 0$. So f is concave up for $|x| > 1$, and f is concave down for $|x| < 1$ by the concavity test. The inflection points are $x = \pm 1$.

8. Find

$$\lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sinh(x)},$$

or prove that the limit doesn't exist.

Solution: We have $\lim_{x\to 0} \frac{x}{\sinh(x)}$ is indeterminate of the form $0/0$, since $\lim_{x \to 0} x = 0$, and $\lim_{x\to 0} \sinh(x) = \sinh(0) = 0$, since both functions are continuous. We take the ratio of the derivatives of the numerator and denominator (using the power rule and the formula for the derivative of $\sinh(x)$): $\lim_{x\to 0} \frac{1}{\cosh(x)} = 1/1 = 1$. Therefore, by l'Hospital's rule, $\lim_{x\to 0} \frac{x}{\sinh(x)} = 1$.

Also, $|x \sin(1/x)| \leq |x|$ for $x \neq 0$ since $|\sin(1/x)| \leq 1$, and thus by the squeeze theorem we have

$$0 = \lim_{x\to 0} -|x| \leq \lim_{x\to 0} x \sin(1/x) \leq \lim_{x\to 0} |x| = 0.$$

Thus, $\lim_{x\to 0} \frac{x}{\sinh(x)} \lim_{x\to 0} x \sin(1/x) = 1 \times 0 = 0 = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin(x)}$ by the product formula for computing limits.