Partial Answers to Review Problems for 2nd Midterm

1. (a) If M is a metric space and $S \subseteq M$, then $S \setminus \partial S$ is open.

   TRUE: If $S \setminus \partial S$ is empty, $S \setminus \partial S$ is open. Otherwise, let $x \in S \setminus \partial S$. $\partial S = \overline{S} \cap \overline{S^c}$ is the set of all $x \in M$ such that every r-neighborhood of x contains points from S and S^c. Since $x \notin \partial S$, there must exist an r-neighborhood $M_r(x)$ of x that is fully contained in S. $M_r(x)$ cannot contain a point from ∂S, since then it would also contain a point from the complement of S.

   (b) $[0,1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$ is a compact subset of $\mathbb{R}$.

   FALSE: By the Heine-Borel Theorem, a subset of $\mathbb{R}$ is compact iff it is closed and bounded. But the given set is not closed since e.g. 1 is a limit point of the set that is not in the set.

   (c) There exists a continuous mapping from $[0,1]$ onto $\mathbb{Q}$.

   FALSE: $[0,1]$ is connected and the images of connected sets under continuous mappings are connected. But $\mathbb{Q}$ is disconnected.

   (d) If $d(x,y)$ is a metric on a set M, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous, strictly increasing function such that $f(0) = 0$ and $f(x+y) \leq f(x) + f(y)$, then $d_f(x,y) = f(d(x,y))$ is also a metric on M.

   The assertion is false if the function is not strictly increasing. For instance, $f(x) \equiv 0$ will not yield a metric. However, if f is strictly increasing, then d_f is a metric. The axioms for a metric are easily verified.

   (e) If a set S contains no interior points (i.e. $\text{int}(S) = \emptyset$), it is closed.

   FALSE: $\mathbb{Q} \subset \mathbb{R}$ has no interior points, but is not a closed subset of $\mathbb{R}$.

   (f) If $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ and a_n converges to $a \in \mathbb{R}$, then $A = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is a compact subset of $\mathbb{R}$.

   TRUE: The sequence is bounded since it is convergent and Cauchy (see p. 75 for how to prove this), and therefore the set A is bounded. Let $(b_m)_{m\in\mathbb{N}}$ be a convergent sequence in the subset. Then for each $m \in \mathbb{N}$, either $b_m = a$ or $b_m = a_{n(m)}$, for some $n(m) \in \mathbb{N}$. Let $b = \lim_{m\to\infty} b_m$. First assume the numbers $n(m)$ are unbounded. For all $\epsilon > 0$, there exists N such that if $n \geq N$, then $d(a_n, a) < \epsilon$, and if $m \geq N$, then $d(b, b_m) < \epsilon$. Since the numbers $n(m)$ are unbounded, we may find $m \geq N$ such that $n(m) \geq N$, and therefore $d(b,a) \leq d(b, a_{n(m)}) + d(a_{n(m)}, a) < 2\epsilon$. Therefore $b = a$, so $b \in A$. Otherwise, there exists N such that $n(m) \leq N$ for all m, and therefore the sequence (b_m) remains in the finite set $\{a, a_1, \ldots, a_N\}$ which is closed, so $b \in A$. We've shown that A is closed and bounded, so by the Heine-Borel theorem, it is compact.

   (g) If a metric space M is connected, then no proper open subset of M is compact.

   TRUE: If M is connected and U is a proper open subset of M, it cannot be closed, since there are no proper clopen subsets in M. But a compact subset of M must be closed and bounded, hence U cannot be compact.

   (h) $\mathbb{R}$ is homeomorphic to $\mathbb{R}^2$.

   FALSE: removing one point disconnects $\mathbb{R}$, but does not disconnect $\mathbb{R}^2$. One may prove that $\mathbb{R}^2$ remains connected upon removing a point using the the same method as problem 59.

(i) The boundary ∂S of a proper subset S of $\mathbb{R}$ is disconnected.

FALSE: Let $S = \{0\}$. Then $\partial S = S$, since every neighborhood of 0 contains 0 and points not in S. But $\{0\}$ is a connected proper subset of $\mathbb{R}$.

2. Suppose $f : K \to \mathbb{R}$ is continuous, where $K \subseteq \mathbb{R}$ is compact. Let the *zero set* of f be defined as

$$Z(f) = \{x \in K : \ f(x) = 0\}.$$

Show that $Z(f)$ is a compact subset of K.

$\{0\}$ is a closed subset of $\mathbb{R}$, $Z(f)$ is the preimage of $\{0\}$ under f, f is continuous, hence $Z(f)$ is closed. A closed subset of a compact set is compact.

3. Let $f : M \to N$ be a continuous bijection and assume M is compact. Show that f is a homeomorphism.

Pugh, Theorem 42, page 81.