Math H185 HW#11

Have you learned enough for grad school? The following questions are from recent prelim exams for first-year grad students. (In the actual exam, one attempts 12 out of 18 questions, and one has to get about 7 out of 12 right to pass.) I will grade this homework on a pass-fail basis (i.e. did you make a reasonable effort or not), and I also hope that it will be helpful in reviewing for the final.

1. Let $U \subset \mathbb{C}$ be simply connected, and suppose $f : U \to \mathbb{C}$ is holomorphic and never zero. Show that there is a holomorphic function $g : U \to \mathbb{C}$ with $e^g = f$.

2. Find the Laurent expansion of
$$f(z) = (1+z)^{-1} + (z^2 - 9)^{-1}$$
in the annulus $1 < |z| < 3$.

3. Let $D \subset \mathbb{C}$ denote the unit disc. If $a, b \in D$, show that there exists a holomorphic bijection $f : D \to D$ with $f(a) = b$.

4. Calculate
$$\int_0^{2\pi} \frac{1}{1 + \frac{1}{2}\sin\theta} d\theta.$$

5. If $0 < r < 1$, find
$$\sum_{k=0}^{\infty} r^k \cos(k\theta).$$
Your final answer should not involve any complex numbers.

6. Find a conformal map from the unit disk $|z| < 1$ to the sector $0 < \arg(z) < \pi/4$.

7. Compute
$$\int_{|z|=2} \frac{z^4}{z^5 - z - 1} dz.$$

8. Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic. Show that if $\mathrm{Re}(f(z)) > \mathrm{Re}(g(z))$ for all z with $|z| = 1$, then $\mathrm{Re}(f(z)) > \mathrm{Re}(g(z))$ for all z with $|z| < 1$.

9. Prove that there are infinitely many complex numbers with $e^z = z$. *Hint:* Consider the behavior of $e^z - z$ on a large square.