Math H185 HW#10

1. Let Ω be an open subset of $\mathbb{C}$, and let $\phi : \Omega \to \mathbb{C}$ be a smooth map (not necessarily holomorphic). If f is a 0-form (i.e. a function) defined on the image of ϕ, define a 0-form $\phi^* f$ on Ω by

$$\phi^* f = f \circ \phi.$$

Let ϕ_1 denote the x-component of ϕ, and let ϕ_2 denote the y-component of ϕ. Define 1-forms $\phi^* dx$ and $\phi^* dy$ on Ω by

$$\begin{aligned} \phi^* dx &= \frac{\partial \phi_1}{\partial x} dx + \frac{\partial \phi_1}{\partial y} dy, \\ \phi^* dy &= \frac{\partial \phi_2}{\partial x} dx + \frac{\partial \phi_2}{\partial y} dy. \end{aligned}$$

If $\alpha = f dx + g dy$ is any 1-form defined on the image of ϕ, define a 1-form $\phi^* \alpha$ on Ω by

$$\phi^* \alpha = (\phi^* f) \phi^* dx + (\phi^* g) \phi^* dy.$$

(a) If f is a 0-form defined on the image of ϕ, show that

$$d(\phi^* f) = \phi^* df.$$

(b) If $\gamma : [a, b] \to \Omega$ is a differentiable arc and α is a 1-form defined on the image of ϕ, show that

$$\int_\gamma \phi^* \alpha = \int_{\phi \circ \gamma} \alpha.$$

(c) Find and prove formulas for $\phi^* dz$ and $\phi^* d\overline{z}$ in terms of $\partial \phi / \partial z$, $\partial \phi / \partial \overline{z}$, dz, and $d\overline{z}$.

2. Given $0 < r < R$, define the annulus

$$A_{r,R} = \{ z \in \mathbb{C} \mid r < |z| < R \}.$$

Prove that there is a holomorphic bijection from $A_{r,R}$ to $A_{r',R'}$ if and only if $R/r = R'/r'$. Hint: to prove the "only if" part, suppose that f is such a bijection and proceed as follows:

(a) Show that if $\{z_n\}$ is a sequence in $A_{r,R}$ with $|z_n|$ converging to r or R, then $|f(z_n)|$ converges to either r' or R'. (If you get stuck, see the top of page 233 of Ahflors.)

(b) Show that in the above situation, the limit of $|f(z_n)|$ is determined by the limit of $|z_n|$.

(c) Use the Schwarz reflection principle (see the top of page 173 in Ahlfors) to extend f to a holomorphic map $\tilde{f}$ between bigger annuli satisfying $\phi' \circ \tilde{f} = \tilde{f} \circ \phi$, where ϕ is reflection in one of the boundary circles of $A_{r,R}$ and ϕ' is reflection in one of the boundary circles of $A_{r',R'}$. (Use some linear fractional transformations to set up the application of the reflection principle.)

(d) Show that $\tilde{f}$ is a bijection between the bigger annuli.

(e) Repeat the previous steps infinitely many times to extend f to a holomorphic bijection $\tilde{f} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$.

(f) Show that there is a constant c such that $\tilde{f}(z) = cz$.