Math 1b Section 2 Midterm #1, 2/14/06, Solutions

1. Substitute $u = \sqrt{e^x+1}$. Then $u^2 = e^x + 1$, so $2u\,du = e^x\,dx = (u^2-1)\,dx$, so $dx = 2u\,du/(u^2-1)$. Thus

$$\int_{\ln 3}^{\ln 8} \frac{dx}{\sqrt{e^x+1}} = \int_2^3 \frac{2\,du}{u^2-1}.$$

To evaluate the integral on the right we use partial fractions:

$$\int_2^3 \frac{2\,du}{u^2-1} = \int_2^3 \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du = \ln\left(\frac{u-1}{u+1}\right)\Big|_2^3 = \ln(3/2).$$

2. The surface area is

$$A = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

We calculate

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2},$$

so

$$A = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{2\pi\left(2+\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}\,dx.$$

To evaluate this integral we substitute $x = \sin\theta$ for $-\pi/2 \le \theta \le \pi/2$. Then $\sqrt{1-x^2} = \cos\theta$ and $dx = \cos\theta\,d\theta$, so

$$A = \int_{-\pi/4}^{\pi/4} 2\pi(2+\cos\theta)\,d\theta = 2\pi^2 + 2\sqrt{2}\pi.$$

3. (a) By definition, the midpoint approximation for $n = 2$ is

$$M_2 = \frac{6-2}{2}\left(\frac{1}{3}+\frac{1}{5}\right) = \frac{16}{15}.$$

(b) If $f(x) = 1/x$ and $x > 0$ then $|f''(x)| = 2/x^3$. This is a decreasing function of x (since its derivative is negative), so if $x \ge 2$ then $|f''(x)| \le 2/2^3 = 1/4 = K$. We use the theorem asserting that the error in the midpoint approximation is bounded by

$$|E_M| \le \frac{K(b-a)^3}{24n^2} = \frac{(1/4)(6-2)^3}{24n^2} = \frac{2}{3n^2}.$$

We want to choose n large enough so that

$$\frac{2}{3n^2} < \frac{1}{100}.$$

We can take $n = 9$ because $2/243 < 1/100$. (Any $n \geq 9$ is guaranteed to work by the theorem. Some smaller values of n might also work but this is not guaranteed by the theorem.)

4. The integral is convergent. To compute it we use partial fractions:

$$\int_2^\infty \frac{dx}{x(x-1)} = \lim_{t\to\infty} \int_2^t \left(\frac{1}{x-1} - \frac{1}{x}\right) dx = \lim_{t\to\infty} \ln\left(\frac{x-1}{x}\right)\Big|_2^t$$
$$= \lim_{t\to\infty} \left(\ln\left(1 - \frac{1}{t}\right) - \ln(1/2)\right) = \ln(2).$$

5. We use integration by parts. We take $u = \arctan x$ and $dv = x^2\,dx$. Then $du = dx/(1+x^2)$ and $v = x^3/3$, so

$$\int x^2 \arctan x\,dx = \frac{x^3 \arctan x}{3} - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx.$$

To evaluate the integral on the right, we divide polynomials to obtain

$$\int \frac{x^3}{1+x^2}\,dx = \int \left(x - \frac{x}{1+x^2}\right) dx = \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C.$$

Putting this all together, we get

$$\int x^2 \arctan x\,dx = \frac{x^3 \arctan x}{3} - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C.$$

6. The integral is divergent, by the comparison test. If $0 < x \leq 1$ then

$$\frac{x + e^x}{x^{3/2}} \geq \frac{e^x}{x^{3/2}} \geq \frac{1}{x^{3/2}} > 0.$$

The second inequality holds because e^x is an increasing function of x (because its derivative is positive), so $e^x \geq e^0$ when $x \geq 0$. We know that

$$\int_0^1 \frac{1}{x^{3/2}}\,dx$$

diverges because $3/2 \geq 1$. So by the comparison test, the integral in question diverges.