Math 113 Homework # 7, selected solutions

Fraleigh 18.12. Let $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Then R is closed under addition because $(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$ and $\mathbb{Q}$ is closed under addition. Indeed $(R, +)$ is a subgroup of $(\mathbb{R}, +)$, since $0 = 0 + 0\sqrt{2} \in R$ and $-(a + b\sqrt{2}) = (-a) + (-b)\sqrt{2}$. R is closed under multiplication because $(a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) = (a_1a_2 + 2b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{2}$. Thus R is a subring of $\mathbb{R}$. Since $\mathbb{R}$ is commutative, so is R. R has a multiplicative identity, namely $1 + 0\sqrt{2}$.

Finally, R is a field, because the multiplicative inverse of $a + b\sqrt{2} \neq 0$ is $(a - b\sqrt{2})/(a^2 - 2b^2)$. Note that the denominator is nonzero because since $\sqrt{2}$ is irrational and a and b are both nonzero, $a - b\sqrt{2} \neq 0$, so the product $a^2 - 2b^2 = (a + b\sqrt{2})(a - b\sqrt{2}) \neq 0$, since $\mathbb{R}$ is an integral domain.

Fraleigh 18.13. This is not a ring since it is not closed under multiplication: $i \cdot i$ is not in the set.

Fraleigh 18.18. An element of a direct product is a unit if and only if each component is a unit. So the units in $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$ are $(\pm 1, x, \pm 1)$ where $x \in \mathbb{Q} \setminus \{0\}$.

Fraleigh 18.28. $x^2 + x - 6 = (x + 3)(x - 2)$. This is zero when $x + 3 = 0$ or $x - 2 = 0$, i.e. $x = -3$ or $x = 2$. But it is also zero if $x + 3$ and $x - 2$ are two nonzero elements whose product is zero, that is if one is ± 2 and the other is 7. This happens when $x = 4, 9$.

Fraleigh 18.32. $\mathbb{Z} \times \mathbb{Z}$ has unity $(1, 1)$, but the subring $(\mathbb{Z} \times \{0\})$ has unity $(1, 0)$.

Fraleigh 18.33. (a) True by definition.

(b) False, e.g. $2\mathbb{Z}$.

(c) False, e.g. $\mathbb{Z}_2$.

(d) False, e.g. $\mathbb{Q}$.

(e) True, e.g. $\mathbb{Z} \subset \mathbb{Q}$.

(f) A matter of opinion, but I would say false. Without the distributive law, the definition of a ring would be stupid, because the addition and multiplication operations would have no relation with each other.

(g) True by definition.

(h) True. More generally, the units in any ring with unity form a group under multiplication. This follows from an argument in a previous homework assignment.

(i) True by definition.

(j) True by definition, since if one takes a ring and considers only the addition operation, then this is a group.

Fraleigh 18.42. Let 1 be the unity in the field and let $1'$ be the unity in the subfield. Since 1 is unity in the field, $1 \cdot 1' = 1'$. Since $1'$ is unity in the subfield, $1' \cdot 1' = 1'$. So $1 \cdot 1' = 1' \cdot 1'$, and since $1' \neq 0$, by definition of a field, we can cancel to get $1 = 1'$.