Math 113 Homework # 6, selected solutions

Fraleigh 14.34. To prove that H is normal, let $a \in G$; we will show that $aHa^{-1} = H$. We know that aHa^{-1} is a subgroup of G with the same cardinality as H, since aHa^{-1} is the image of H under the isomorphism $i_a : G \to G$. Since we assumed that G has only one subgroup of this cardinality, it follows that $aHa^{-1} = H$.

Fraleigh 14.37. This problem adds a layer of abstraction to what we have usually been doing, because now automorphisms of a group are being made into a new, more abstract group. A similar process of abstraction happens frequently in mathematics. But once you understand what the problem means, it is not too hard to solve.

(a) Let $\mathrm{Aut}(G)$ denote the set of automorphisms of G. We need to show that this is a group under composition. We know that this is closed under composition because from previous homework problems, the composition of two bijections is a bijection and the composition of two homomorphisms is a homomorphism. We know that composition of functions is associative. The identity map $\mathrm{id}_G : G \to G$ is an automorphism which serves as the identity element in the group $\mathrm{Aut}(G)$, since composing any function with the identity map gives the same function. Finally every element has an inverse, because if $f \in \mathrm{Aut}(G)$ then we know that the inverse function $f^{-1} \in \mathrm{Aut}(G)$, and f^{-1} is the group-theoretic inverse of f because $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}_G$ by the definition of the inverse function.

(b) Let $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$ denote the set of inner automorphisms of G. To see that this is a subgroup, we note that it contains the identity because $\mathrm{id}_G = i_e$; it is closed under composition because $i_a \circ i_b = i_{ab}$, since

$$i_{ab}(x) = abx(ab)^{-1} = a(bxb^{-1})a^{-1} = i_a(i_b(x)); \tag{1}$$

and it is closed under inverses because equation (1) implies that $(i_a)^{-1} = i_{a^{-1}}$.

More elegantly, equation (1) implies that the map $G \to \mathrm{Aut}(G)$ sending $a \mapsto i_a$ is a homomorphism, and $\mathrm{Inn}(G)$ is its image, hence a subgroup of $\mathrm{Aut}(G)$, since homomorphisms send subgroups to subgroups.

To prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, let $\phi \in \mathrm{Aut}(G)$ and $i_a \in \mathrm{Inn}(G)$; we will show that $\phi \circ i_a \circ \phi^{-1} \in \mathrm{Inn}(G)$. In fact we have $\phi \circ i_a \circ \phi^{-1} = i_{\phi(a)}$, because

$$\phi(i_a(\phi^{-1}(x))) = \phi(a\phi^{-1}(x)a^{-1}) = \phi(a)\phi(\phi^{-1}(x))\phi(a^{-1}) = \phi(a)x\phi(a)^{-1}.$$

Here the first equality uses the definition of i_a, and the last two equalities use the fact that ϕ is a homomorphism.

Fraleigh 15.14. You can prove that for any groups G and H, $Z(G \times H) = Z(G) \times Z(H)$ and $C(G \times H) = C(G) \times C(H)$. Then $Z(\mathbb{Z}_3 \times S_3) = \mathbb{Z}_3 \times \{e\}$ and $C(\mathbb{Z}_3 \times S_3) = \{0\} \times A_3$. (Here $C(G)$ denotes the commutator subgroup of G; a more common notation for this is $[G, G]$.)

Fraleigh 15.19. (a) True. If G is cyclic with generator a, then for any subgroup H of G, the coset of a generates G/H.

(b) False. Trivial example: if G is noncyclic then G/G is cyclic. Nontrivial example: S_3 is not cyclic but S_3/A_3 is cyclic.

(c) False. The coset of $1/2$ has order 2 in $\mathbb{R}/\mathbb{Z}$.

(d) True, the coset of $1/n$ has order n.

(e) False. The only elements of order 4 are the cosets of $1/4$ and $3/4$.

(f) True. If $C(G) = \{e\}$ then for all $x, y \in G$ we have $xyx^{-1}y^{-1} = e$, i.e. $xy = yx$.

(g) False. For example $\mathbb{Z}_4/\langle 2 \rangle$ is abelian but the commutator subgroup of $\mathbb{Z}_4$ does not contain 2. (It is however true that if G/H is abelian then the commutator subgroup of G is contained in H.)

(h) False. $\mathbb{Z}_2$ is simple but its commutator subgroup is $\{0\}$.

(i) True. If G is simple then the commutator subgroup, being normal, must be $\{e\}$ or G, and the former possibility is excluded because G is nonabelian.

(j) False. A_5 is simple and has order 60 which is not prime.

4. Before starting, recall the multiplication rules for the dihedral group: $R_iR_j = R_{i+j}$, $R_iF_j = F_{i+j}$, $F_iR_j = F_{i-j}$, and $F_iF_j = R_{i-j}$. These imply that $F_iR_jF_i^{-1} = R_{-j}$. (Can you understand this geometrically?)

(a) Let $R_{kd} \in H$; we must show that $aR_{kd}a^{-1} \in H$ for all $a \in D_n$. If $a = R_i$ then $aR_{kd}a^{-1} = R_{kd} \in H$ since all rotations commute. If $a = F_i$ then the above calculation shows that $aR_{kd}a^{-1} = R_{-kd} \in H$.

(b) For each $i \in \{0, \ldots, d-1\}$ there are two cosets

$$\begin{aligned} R_i H &= \{R_i, R_{i+d}, R_{i+2d}, \ldots\}, \\ F_i H &= \{F_i, F_{i+d}, F_{i+2d}, \ldots\}. \end{aligned}$$

These are all the cosets. One can see this because every element of D_n is in one of the above cosets. Alternatively, note that since $|D_n| = 2n$ and $|H| = n/d$, there are $2d$ cosets, and we have found $2d$ distinct cosets above.

(c) The multiplication rules for D_n involve adding and subtracting the subscripts 'i' and 'j' mod n. The multiplication rules for D_d are the same except that we do everything mod d. So if we identify R_i or F_i in D_d with the coset $R_i H$ or $F_i H$ in D_n, then this is an isomorphism.

(d) Using the multiplication rules we compute all possible commutators: $[R_i, R_j] = R_0$, $[R_i, F_j] = R_{2i}$, $[F_j, R_i] = R_{-2i}$, and $[F_i, F_j] = R_{2(i-j)}$. Thus the set of commutators is the set $\{R_0, R_2, R_4, \ldots\}$. The commutator subgroup is the subgroup generated by all commutators. In this example the set of all commutators is already a subgroup, so it is the commutator subgroup.

(e) By parts (c) and (d), $D_{2n}^{ab} = D_{2n}/\{R_0, R_2, R_4, \ldots\} \simeq D_2$. We showed in class that $D_2 \simeq V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Remark: if n is odd then $D_n^{\mathrm{ab}} \simeq \mathbb{Z}_2$. Can you see why?