Math 113 Homework # 4, selected solutions

2:

(a) Let $i \in \{1, \ldots, n\}$; we must show that $f(g(i)) = g(f(i))$.

Case 1: i is not in the support of f or g. Then $f(g(i)) = f(i) = i$ and likewise $g(f(i)) = g(i) = i$.

Case 2: i is in the support of f. Then $f(i)$ is also in the support of f, since otherwise f would send both i and $f(i)$ to $f(i)$, and hence would not be injective since $i \neq f(i)$. Since i is not in the support of g we have $f(g(i)) = f(i)$, and since $f(i)$ is not in the support of g we have $g(f(i)) = f(i)$.

Case 3: i is in the support of g. By the same argument as in Case 2 but with f and g switched, we have $f(g(i)) = g(f(i)) = g(i)$.

(b) Let p be the element of $\mathrm{supp}(f) \cap \mathrm{supp}(g)$. Then you can check that $[f, g](p) = f(p)$, $[f, g](f(p)) = g(p)$, $[f, g](g(p)) = p$, and for any $x \in \{1, \ldots, n\} \setminus \{p, f(p), g(p)\}$, we have $[f, g](x) = x$. Also p, $f(p)$, and $g(p)$ are all distinct (why?), so $[f, g]$ is the 3-cycle

$$[f, g] = (p,\ g(p),\ f(p)).$$

3:

(a) Let $i \in \{1, \ldots, n\}$; we need to show that

$$\sigma\mu\sigma^{-1}(i) = (\sigma(x_1)\ \sigma(x_2)\ \cdots\ \sigma(x_k))(i).$$

We consider two cases.

Case 1: $i = \sigma(x_j)$ for some j. Then

$$\sigma\mu\sigma^{-1}(i) = \sigma\mu(x_j) = \sigma(x_{j+1}) = (\sigma(x_1)\ \sigma(x_2)\ \cdots\ \sigma(x_k))(i).$$

Here it is understood that we compute $j + 1 \bmod k$.

Case 2: $i \notin \{\sigma(x_1), \ldots, \sigma(x_k)\}$. Then $\sigma^{-1}(i) \notin \{x_1, \ldots, x_k\}$, so

$$\sigma\mu\sigma^{-1}(i) = \sigma(\sigma^{-1}(i)) = i = (\sigma(x_1)\ \sigma(x_2)\ \cdots\ \sigma(x_k))(i).$$

(b) Two cycles $x, y \in S_n$ are conjugate if and only if the lengths of the cycles in the disjoint cycle factorization of x are the same as the lengths of the cycles in the disjoint cycle factorization of y.

4: We show by induction on $n > 1$ that any permutation in A_n is a product of $\leq n-2$ three-cycles. This follows the same strategy as our proof in class that any permutation is a product of transpositions. The base case $n = 2$ holds because the only element of A_2 is the identity which is the product of 0 three-cycles. Now let $n > 2$, assume the result holds for $n-1$, and let $f \in A_n$. If $f(n) = n$, let f' be the restriction of f to the set $\{1, \ldots, n-1\}$. Then $f' \in A_{n-1}$ so by inductive hypothesis, f', and hence f, is the product of $\leq n-3$ three-cycles. If $f(n) \neq n$, then since $n \geq 3$, we can find a three-cycle σ such that $\sigma(f(n)) = n$. Namely pick $i \in \{1, \ldots, n\}$ distinct from n and $f(n)$ and let $\sigma = (f(n), n, i)$. Then $(\sigma \circ f)(n) = n$, and $\sigma \circ f \in A_n$ since a three-cycle is an even permutation. So by inductive hypothesis as above, $\sigma \circ f$ is a product of $\leq n-3$ three-cycles, and then multiplying on the left by the three-cycle σ^{-1}, we get that f is a product of $\leq n-2$ three-cycles.

For another proof, see the hint in section 15, exercise 39b.

5: If we number the diagonals of the cube $1, 2, 3, 4$, then we have a map $\phi : G \to S_4$ which sends a symmetry of the cube to the induced permutation of the diagonals. We claim that ϕ is an isomorphism. We have $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$ since the group operations in both G and S_4 are given by composition of functions. So we just have to show that ϕ is bijective. Now $|G| = 24$ as shown in class and $|S_4| = 4! = 24$, so since the domain and codomain of ϕ have the same (finite) cardinality it is enough to show that ϕ is surjective.

Let $\sigma \in S_4$. We need to find a rotation of the cube effecting the permutation σ on the diagonals. Rotating the cube around an axis passing through two opposite faces cycles the four axes around, so by rotating the cube this way we can send diagonal 1 to $\sigma(1)$. Now we have to permute the three diagonals other than $\sigma(1)$ to put them into the correct places. By rotating the cube around the diagonal $\sigma(1)$, we can cycle around the other three diagonals. This gives us half of the six permutations of the other three diagonals. To get the other three permutations we have to rotate the cube so as to flip the diagonal $\sigma(1)$. Namely, rotating the cube 180 degrees around an axis orthogonal to the diagonals $\sigma(1)$ and i, where $i \neq \sigma(1)$, will switch the other two diagonals. (You should try this with your Rubik's cube.)

Remarks: 1) Instead of proving that ϕ is surjective, one could instead prove that ϕ is injective, i.e. that a symmetry of the cube is uniquely determined by its action on the diagonals. This isn't obvious and requires geometric arguments similar to the previous paragraph. (One has to be careful to use the fact that we are not allowing reflections. The reflection across the center point of the cube induces the identity permutation on the diagonals!)

2) This problem does *not* follow from Cayley's theorem. Cayley's theorem says that any finite group G is isomorphic to a subgroup of S_n for some n. But n might have to be really big; the proof needs to take $n = |G|$. (This theorem isn't terribly useful for the topics in this course, which is one of the reasons why we haven't covered it yet.)

6: We just read this off from the multiplication table.

(a) The left cosets of $H = \{R_0, F_0\}$ are $\{R_0, F_0\}$, $\{R_1, F_1\}$, $\{R_2, F_2\}$, and $\{R_3, F_3\}$. The right cosets of H are $\{R_0, F_0\}$, $\{R_1, F_3\}$, $\{R_2, F_2\}$, and $\{R_3, F_1\}$. These are not the same as the left cosets.

(b) The left cosets of $H = \{R_0, R_2\}$ are $\{R_0, R_2\}$, $\{R_1, R_3\}$, $\{F_0, F_2\}$, and $\{F_1, F_3\}$. The right cosets of H are the same as the left cosets.