

## Math 113 Homework # 2, selected solutions

Fraleigh 3.26: Suppose that  $\phi : (S, *) \rightarrow (S', *')$  is an isomorphism of binary structures; prove that  $\phi^{-1}$  is an isomorphism of binary structures.

Since  $\phi \circ \phi^{-1} = \text{id}_{S'}$  and  $\phi^{-1} \circ \phi = \text{id}_S$  and the identity is bijective, it follows from a basic lemma from class that  $\phi^{-1}$  is a bijection. So we just have to check that  $\phi^{-1}(x *' y) = \phi^{-1}(x) * \phi^{-1}(y)$  for all  $x, y \in S'$ . Since  $\phi$  is injective, it is enough to check that  $\phi$  of the left side of this equation equals  $\phi$  of the right side, i.e.

$$\phi(\phi^{-1}(x *' y)) = \phi(\phi^{-1}(x) * \phi^{-1}(y)). \quad (1)$$

By the definition of  $\phi^{-1}$ , the left side of equation (1) is

$$\phi(\phi^{-1}(x *' y)) = x *' y.$$

Since  $\phi$  is an isomorphism and by the definition of  $\phi^{-1}$ , the right side of equation (1) is

$$\begin{aligned} \phi(\phi^{-1}(x) * \phi^{-1}(y)) &= \phi(\phi^{-1}(x)) *' \phi(\phi^{-1}(y)) \\ &= x *' y. \end{aligned}$$

Hence equation (1) holds.

Fraleigh 4.19: (a) Suppose  $a, b \in \mathbb{R} \setminus \{-1\}$ ; we need to check that  $a * b \in \mathbb{R} \setminus \{-1\}$ . Since  $\mathbb{R}$  is closed under addition and multiplication,  $a * b \in \mathbb{R}$ , so we just have to check that  $a + b + ab \neq -1$ . Suppose to the contrary that  $a + b + ab = -1$ . Then  $a + b + ab + 1 = 0$ , i.e.  $(a + 1)(b + 1) = 0$ , so  $a + 1 = 0$  or  $b + 1 = 0$ , i.e.  $a = -1$  or  $b = -1$ , a contradiction.

(b), (c). This is pretty straightforward, but the following clever observation makes it much easier. Namely, the equation

$$a * b + 1 = (a + 1)(b + 1)$$

shows that  $\phi : x \mapsto x + 1$  defines an isomorphism of binary structures from  $(\mathbb{R} \setminus \{-1\}, *)$  to  $(\mathbb{R} \setminus \{0\}, \cdot)$ . Hence  $(\mathbb{R} \setminus \{-1\}, *)$  is a group. The solution to  $2 * x * 3 = 7$  is  $\phi^{-1}$  of the solution to  $3 \cdot y \cdot 4 = 8$ , so  $y = 2/3$  and  $x = -1/3$ . (Can you fill in the details of the preceding?)

- 3: (a)  $\mathbb{Z}_n^*$  is closed under multiplication because if  $x_1$  and  $x_2$  are units then there exist  $y_1, y_2 \in \mathbb{Z}_n$  with  $x_1 y_1 = x_2 y_2 = 1$ , so since multiplication in  $\mathbb{Z}_n$  is commutative,  $(x_1 x_2)(y_1 y_2) = 1$ , so  $x_1 x_2$  is a unit. Multiplication on  $\mathbb{Z}_n^*$  is associative because multiplication on  $\mathbb{Z}_n$  is associative (because multiplication on integers is associative); 1 is an identity in  $\mathbb{Z}_n^*$  (1 is a unit since  $1 \cdot 1 = 1$ ); and every element of  $\mathbb{Z}_n^*$  has an inverse by definition.
- (b), (c) As sets,  $\mathbb{Z}_8^* = \{1, 3, 5, 7\}$ ,  $\mathbb{Z}_{10}^* = \{1, 3, 7, 9\}$ , and  $\mathbb{Z}_{12}^* = \{1, 5, 7, 11\}$ . I won't draw the multiplication tables. But you can see by inspection that  $\mathbb{Z}_8^*$  and  $\mathbb{Z}_{12}^*$  both have the property that every element is its own inverse, and the product of any two nonidentity elements is the third. This property completely characterizes the multiplication rule and shows that  $\mathbb{Z}_8^*$  and  $\mathbb{Z}_{12}^*$  are both isomorphic to the Klein four-group  $V_4$ . On the other hand  $\mathbb{Z}_{10}^*$  is not isomorphic to the Klein four-group since 3 and 7 are not equal to their own inverses. (Indeed  $\mathbb{Z}_{10}^* \simeq \mathbb{Z}_4$ .)