

Math 113 Homework # 1, selected solutions

Keep in mind that most true statements have many correct proofs.

2a. Suppose $g \circ f$ is surjective. We need to show that g is surjective. Let $z \in Z$. We need to show that there exists $y \in Y$ such that $g(y) = z$. Since $g \circ f$ is surjective, there exists $x \in X$ such that $(g \circ f)(x) = z$. Then $g(f(x)) = z$, so we can take $y = f(x)$.

Suppose $g \circ f$ is injective. We need to show that f is injective. Suppose $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$; we need to show that $x_1 = x_2$. Applying g to the equation $f(x_1) = f(x_2)$, we obtain $g(f(x_1)) = g(f(x_2))$. Equivalently, $(g \circ f)(x_1) = (g \circ f)(x_2)$. Since $g \circ f$ is injective, it follows that $x_1 = x_2$.

2b. (\Rightarrow) Suppose $f : X \rightarrow Y$ is bijective. We need to show that there exists $g : Y \rightarrow X$ with $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. Define $g : Y \rightarrow X$ as follows: if $y \in Y$, then $g(y)$ is the unique $x \in X$ such that $f(x) = y$.

To show that $g \circ f = \text{id}_X$, let $x \in X$; we need to show that $g(f(x)) = x$. By definition, $g(f(x))$ is the unique $x' \in X$ such that $f(x') = f(x)$. It follows that $x' = x$.

To show that $f \circ g = \text{id}_Y$, let $y \in Y$; we need to show that $f(g(y)) = y$. By definition $g(y)$ is the unique $x \in X$ such that $f(x) = y$. Thus $f(g(y)) = y$.

(\Leftarrow) Suppose $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. We need to show that f is bijective. Of course, id_X and id_Y are bijective. Since $g \circ f$ is injective, f is injective by part (a). Since $f \circ g$ is surjective, f is surjective by part (a).

2c. *Informal proof.* (\Rightarrow) Suppose f is injective. Then no two elements of X get sent to the same element of Y . Hence the image of f , i.e. the set of all elements in Y that are f of something, has at least the same cardinality as X . Since $|X| = |Y|$, it follows that the image of f is all of Y , so f is surjective.

(\Leftarrow) Suppose f is surjective. Then f must be injective, because otherwise two different elements of X would be sent to the same element of Y , so the image of f would have cardinality less than that of X , so since $|X| = |Y|$ it would follow that f is not surjective, a contradiction.

Formal proof. To make the above more precise, the following preliminary observation is useful. For each $y \in Y$, let $n(y)$ denote the number of

preimages of y , i.e. elements $x \in X$ with $f(x) = y$. We observe that

$$\sum_{y \in Y} n(y) = |X|. \quad (1)$$

The reason is that the sum on the left is the total number of preimages of all elements of Y ; and this equals $|X|$ because every element $x \in X$ is the preimage of exactly one element of Y , namely $f(x)$.

(\Rightarrow) Suppose f is injective. Then $n(y) \leq 1$ for all $y \in Y$. We must then have $n(y) = 1$ for all $y \in Y$, or else the left side of (1) would be smaller than $|Y|$, contradicting our assumption that $|X| = |Y|$. Therefore f is surjective.

(\Leftarrow) Suppose f is surjective. Then $n(y) \geq 1$ for all $y \in Y$. We must then have $n(y) = 1$ for all $y \in Y$, or else the left side of (1) would be bigger than $|Y|$, contradicting our assumption that $|X| = |Y|$. Therefore f is injective.

3. Suppose $x \equiv x' \pmod{n}$ and $y \equiv y' \pmod{n}$. We must show that $xy \equiv x'y' \pmod{n}$. We can write $x' = x + kn$ and $y' = y + ln$ where k and l are integers. Then

$$\begin{aligned} x'y' &= (x + kn)(y + ln) \\ &= xy + (xl + ky + kln)n, \end{aligned}$$

so $x'y' \equiv xy \pmod{n}$.

4a. *Base case.* when $n = 1$ this is true since $1 = 1^2$.

Inductive step. Let n be a positive integer and suppose that

$$1 + 3 + \cdots + (2n - 1) = n^2.$$

We must show that

$$1 + 3 + \cdots + (2n - 1) + (2n + 1) = (n + 1)^2.$$

This follows from the inductive hypothesis since $n^2 + (2n + 1) = (n + 1)^2$.

To see this without induction, one can cut up an $n \times n$ grid into “shells” whose areas are the first n odd numbers (try this). Alternatively, one can observe that since the numbers $1, 3, \dots, 2n-1$ are evenly spaced, their average is $(1 + (2n - 1))/2 = n$, and their sum is the number of numbers, namely n , times the average.

4b. Show by induction on n that if n is a nonnegative integer and x is a real number with $x \neq 1$ then

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}. \quad (2)$$

Base case. This is true when $n = 0$ since $1 = (1 - x^{0+1})/(1 - x)$.

Inductive step. Suppose that (2) holds for a given nonnegative integer n . Then

$$\begin{aligned} 1 + x + x^2 + \cdots + x^n + x^{n+1} &= (1 + x + \cdots + x^n) + x^{n+1} \\ &= \frac{1 - x^{n+1}}{1 - x} + x^{n+1} \\ &= \frac{1 - x^{n+1}}{1 - x} + \frac{x^{n+1}(1 - x)}{1 - x} \\ &= \frac{1 - x^{n+2}}{1 - x}. \end{aligned}$$

To prove (2) without using induction, we note that since $1 - x \neq 0$, the equation (2) is true if and only if it is true after both sides are multiplied by $1 - x$, i.e.

$$(1 - x)(1 + x + x^2 + \cdots + x^n) = 1 - x^{n+1}.$$

Multiplying out the left side, we see that this is equivalent to

$$(1 + x + x^2 + \cdots + x^n) - (x + x^2 + x^3 + \cdots + x^{n+1}) = 1 - x^{n+1}.$$

But this is clearly true since on the left side, all terms cancel in pairs except for 1 and $-x^{n+1}$.

5. We use induction on n .

Base case. If $n = 1$, this is true, since a 2×2 checkerboard with one square removed is itself an L -triomino and so can be tiled using one L -triomino.

Inductive step. Suppose the claim is true for n , and consider a $2^{n+1} \times 2^{n+1}$ checkerboard with one square removed. We can divide the $2^{n+1} \times 2^{n+1}$ checkerboard into four quadrants of size $2^n \times 2^n$. Around the center of the $2^{n+1} \times 2^{n+1}$ checkerboard, there are four “center squares”, each of which is a corner of one of the quadrants. The removed square is in one quadrant, and we temporarily remove the three center squares corresponding to the other three quadrants. Now each quadrant is missing a square and so can be tiled by inductive hypothesis. To complete the tiling we place a final L -triomino over the three center squares that we removed. (Try an example with $n = 8$.)