

The embedded contact homology index revisited

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Abstract

Let Y be a closed oriented 3-manifold with a contact form such that all Reeb orbits are nondegenerate. The embedded contact homology (ECH) index associates an integer to each relative 2-dimensional homology class of surfaces whose boundary is the difference between two unions of Reeb orbits. This integer determines the relative grading on ECH; the ECH differential counts holomorphic curves in the symplectization of Y whose relative homology classes have ECH index 1. A known index inequality implies that such curves are (mostly) embedded and satisfy some additional constraints.

In this paper we prove four new results about the ECH index. First, we refine the relative grading on ECH to an absolute grading, which associates to each union of Reeb orbits a homotopy class of oriented 2-plane fields on Y . Second, we extend the ECH index inequality to symplectic cobordisms between three-manifolds with stable Hamiltonian structures, and simplify the proof. Third, we establish general inequalities on the ECH index of unions and multiple covers of holomorphic curves in cobordisms. Finally, we define a new relative filtration on ECH, or any other kind of contact homology of a contact 3-manifold, which is similar to the ECH index and related to the Euler characteristic of holomorphic curves. This does not give new topological invariants except possibly in special situations, but it is a useful computational tool.

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1 Introduction

We begin with a very brief overview of embedded contact homology, and then describe the results of this paper. More detailed definitions will be given later.

1.1 Embedded contact homology

Let Y be a closed oriented 3-manifold with a contact form λ such that all Reeb orbits are nondegenerate. Let $\xi := \text{Ker}(\lambda)$ denote the associated contact structure, and let $\Gamma \in H_1(Y)$. The embedded contact homology $ECH_*(Y, \xi, \Gamma)$ is the homology of a chain complex which is generated by finite sets of pairs $\alpha = \{(\alpha_i, m_i)\}$, where the α_i 's are distinct embedded Reeb orbits, the m_i 's are positive integers, $\sum_i m_i [\alpha_i] = \Gamma \in H_1(Y)$, and $m_i = 1$ whenever α_i is hyperbolic. The differential ∂ on the chain complex counts certain (mostly) embedded holomorphic curves in $\mathbb{R} \times Y$, with respect to a suitable \mathbb{R} -invariant almost complex structure J .

More precisely, the differential counts holomorphic curves C whose ECH index equals one. The ECH index $I(C)$, originally defined in [11] and reviewed here in §2, is a certain topological quantity which depends only on the relative homology class of C . The relation between the condition $I(C) = 1$ and embeddedness is as follows. It is shown¹ in [11] that if C is not multiply covered, then

$$\text{ind}(C) \leq I(C) - 2\delta(C), \quad (1.1)$$

where $\delta(C)$ is a nonnegative integer which equals zero if and only if C is embedded. Here $\text{ind}(C)$ denotes the Fredholm index of C , which is the dimension of the moduli space of holomorphic curves near C if J is generic. It is further shown in [11] that if T is a union of \mathbb{R} -invariant cylinders, and if the image of C contains no \mathbb{R} -invariant cylinder, then

$$I(C \cup T) \geq I(C) + 2\#(C \cap T), \quad (1.2)$$

where ‘ $\#$ ’ denotes the algebraic intersection number, which is nonnegative by intersection positivity. The inequalities (1.1) and (1.2) imply, as explained in [13, Cor. 11.5], that if J is generic, then any holomorphic curve C with $I(C) = 1$ consists of an embedded component of Fredholm index one, possibly together with some covers of \mathbb{R} -invariant cylinders which do not intersect the rest of C . These are the curves that the ECH differential counts. In particular, I defines the relative grading on the chain complex. See [13] for more about ECH, and [14, §7] for a proof that $\partial^2 = 0$.

A priori, ECH might depend not only on Y , ξ , and Γ , but also on the choice of contact form λ and almost complex structure J . However, Taubes [25] has recently shown that, as conjectured in [13], ECH is not only independent of λ and J , but also isomorphic to a version of Seiberg-Witten Floer homology as defined by Kronheimer-Mrowka [16]. The precise statement is

¹The index inequality (1.1) was proved in a different and easier context in [11]. To carry over the argument to the present setting, one needs to use the asymptotic analysis of Siefring [21], see §4.

that²

$$ECH_*(Y, \xi, \Gamma) \simeq \check{H}M_*(-Y, \mathfrak{s}(\xi) + \text{PD}(\Gamma)), \quad (1.3)$$

up to a grading shift, where $\mathfrak{s}(\xi)$ is a spin-c structure determined by ξ , see §3.1. Thus ECH is more or less a topological invariant of the three-manifold Y , and in this regard it differs substantially from the symplectic field theory of Eliashberg-Givental-Hofer [5, 6], which is highly sensitive to the contact structure and vanishes for overtwisted ones [2, 28]. The isomorphism (1.3) can be regarded as an extension of Taubes’s “Seiberg-Witten=Gromov” theorem for closed symplectic 4-manifolds [24] to the noncompact symplectic 4-manifold $\mathbb{R} \times Y$. This hoped-for correspondence was the original motivation for the definition of ECH, see [11].

1.2 New results on the ECH index

Despite this motivation, the definition of ECH, and especially the ECH index, may at first seem a bit strange. The aim of this paper is to shed some additional light on the ECH index by proving four new results about it.

1.2.1 Absolute grading

First, in §3 we show that the relative grading on ECH can be refined to an absolute grading, which associates to each generator a homotopy class of oriented 2-plane fields on Y , see Theorem 3.1. If $\alpha = \{(\alpha_i, m_i)\}$ is an ECH generator, then the associated 2-plane field $I(\alpha)$ is obtained by modifying the contact plane field ξ in a canonical manner (up to homotopy, depending only on m_i) in disjoint tubular neighborhoods of the Reeb orbits α_i .

Recall from [16] that Seiberg-Witten Floer homology also has an absolute grading by homotopy classes of oriented 2-plane fields. We conjecture that Taubes’s isomorphism (1.3) between ECH and Seiberg-Witten Floer homology respects these absolute gradings.

We also expect that one can define a similar absolute grading on Heegaard Floer homology, by refining the construction in [19, §2.6] that associates to each Heegaard-Floer generator a spin-c structure.

1.2.2 Index inequality in cobordisms

Second, in §4 we generalize the index inequality (1.1) to holomorphic curves in four-dimensional symplectic cobordisms, see Theorem 4.15. Our proof follows the original proof of (1.1), but with a new and simpler proof of the key combinatorial lemma.

²Taubes replaces the r.h.s. of (1.3) with the isomorphic group $\widehat{HM}^{-*}(Y, \mathfrak{s}(\xi) + \text{PD}(\Gamma))$. This is also isomorphic to the completed version $\check{H}M_\bullet(-Y, \mathfrak{s}(\xi) + \text{PD}(\Gamma))$, and conjecturally isomorphic to the Heegaard Floer homology $HF_*^+(-Y, \mathfrak{s}(\xi) + \text{PD}(\Gamma))$.

1.2.3 Unions and multiple covers

Third, in §5 we prove a new inequality on the ECH index of unions and multiple covers of holomorphic curves in cobordisms, see Theorem 5.1. This inequality is a substantial generalization of (1.2) and asserts that if C and C' are two holomorphic curves, then

$$I(C \cup C') \geq I(C) + I(C') + 2C \cdot C', \quad (1.4)$$

where $C \cdot C'$ is an “intersection number” of C and C' defined in §5.1. If the images of C and C' do not have any irreducible components in common, then $C \cdot C'$ is simply the algebraic count of intersections of C and C' , which is nonnegative by intersection positivity. If the images of C and C' have a common component, then the definition of $C \cdot C'$ is more subtle. In particular, $C \cdot C$ can be negative.

Ultimately, when X is a symplectic cobordism from Y_+ to Y_- , one would like to define a map from the ECH of Y_+ to that of Y_- by counting holomorphic curves C in X with ECH index $I(C) = 0$. A major difficulty is that even if J is generic, an arbitrary $I = 0$ curve may contain some negative ECH index multiple covers, together with some other components with positive Fredholm index. The inequality (1.4) clarifies the extent to which this can happen. Note that this problem does not arise in defining the ECH differential. Indeed, if X is a symplectization $\mathbb{R} \times Y$ with an \mathbb{R} -invariant almost complex structure, then with some trivial exceptions $C \cdot C$ is always nonnegative, see Proposition 5.6.

1.2.4 Euler characteristic and relative filtration

While the ECH differential counts holomorphic curves C with $I(C) = 1$, the latter condition does not specify the genus or Euler characteristic of C . To complete the picture here, the last part of this paper introduces another relative index, which we denote by J_0 . This is a natural cousin of the ECH index I , and has similar basic properties. An analogue of the inequality (1.1) holds for J_0 , in which J_0 bounds the negative Euler characteristic instead of the Fredholm index, see Corollary 6.10 and the stronger Proposition 6.9. A version of the inequality (1.4) also holds for J_0 , see Proposition 6.14. The resulting bound on the topological complexity of holomorphic curves in terms of J_0 plays a key role in a subsequent paper [15], which obtains various extensions of the Weinstein conjecture.

The above inequalities also lead to the last main result of the present paper, Theorem 6.6, asserting that if X is the symplectization of a contact manifold Y with an \mathbb{R} -invariant almost complex structure, then every holomorphic curve C in X satisfies $J_+(C) \geq 0$. Here J_+ is another relative index which is a slight variant of J_0 . It follows that J_+ defines a relative

filtration on embedded contact homology, or for that matter on any kind of contact homology of a contact 3-manifold. As explained in §6.2, this filtration is a useful computational tool, although it does not give new topological invariants except possibly in special situations.

1.2.5 Stable Hamiltonian structures

Embedded contact homology is very similar to the periodic Floer homology (PFH) of mapping tori considered in [11, 12]. In fact, there is a more general geometric structure from [1], called a “stable Hamiltonian structure”, which includes both contact manifolds and mapping tori as special cases, and for which one still has Gromov-type compactness for holomorphic curves. The definition of ECH or PFH then extends in a straightforward way to any 3-manifold with a stable Hamiltonian structure in which all Reeb orbits are nondegenerate³. For this reason, we will use stable Hamiltonian structures as the basic geometric setup throughout this paper.

2 The ECH index

We now review the definition of the ECH index, and the various notions that enter into it, in the context of stable Hamiltonian structures.

2.1 Stable Hamiltonian structures

Let Y be an oriented 3-manifold. For simplicity we assume that Y is closed, although for most of this paper this is not actually necessary.

Definition 2.1. [1, 3, 21] A **stable Hamiltonian structure** on Y is a pair (λ, ω) , where λ is a 1-form on Y , and ω is a 2-form on Y , such that:

$$\begin{aligned}\lambda \wedge \omega &> 0, \\ d\omega &= 0, \\ \text{Ker}(\omega) &\subset \text{Ker}(d\lambda).\end{aligned}$$

A stable Hamiltonian structure determines an oriented 2-plane field

$$\xi := \text{Ker}(\lambda).$$

It also determines a vector field R defined by

$$\omega(R, \cdot) = 0, \quad \lambda(R) = 1.$$

³When the stable Hamiltonian structure is not contact, one needs to either assume a “monotonicity” condition as in [12, §2], or work with coefficients in a suitable Novikov ring.

We will call R the **Reeb vector field**, and the flow determined by R the **Reeb flow**. The definition of stable Hamiltonian structure implies that R is transverse to ξ , the restriction of ω to ξ is nondegenerate, and the Reeb flow preserves the stable Hamiltonian structure, i.e. $\mathcal{L}_R\lambda = 0$ and $\mathcal{L}_R\omega = 0$.

Example 2.2. If λ is a contact 1-form on Y , i.e. $\lambda \wedge d\lambda > 0$, then $(\lambda, d\lambda)$ is a stable Hamiltonian structure, in which ξ is the contact 2-plane field, and R is the Reeb vector field in the usual sense.

Example 2.3. Let Σ be a surface with a symplectic form ω , and let $\phi : (\Sigma, \omega) \rightarrow (\Sigma, \omega)$ be a symplectomorphism. Let Y be the **mapping torus**

$$Y := \frac{[0, 1] \times \Sigma}{(1, x) \sim (0, \phi(x))}.$$

Projection onto the $[0, 1]$ factor defines a fiber bundle $\pi : Y \rightarrow S^1$. Let t denote the $[0, 1]$ coordinate. The vector field ∂_t on $[0, 1] \times \Sigma$ descends to a vector field on Y , which we also denote by ∂_t . The 2-forms ω on the fibers of Y extend to a closed 2-form ω_Y on Y which annihilates ∂_t . Then (π^*dt, ω_Y) is a stable Hamiltonian structure on Y , in which ξ is the vertical tangent bundle of π , and $R = \partial_t$.

2.2 Reeb orbits

Fix a closed oriented 3-manifold Y with a stable Hamiltonian structure (λ, ω) . A **Reeb orbit** is a closed orbit of the Reeb flow, i.e. a smooth map $\gamma : \mathbb{R}/T \rightarrow Y$ for some $T > 0$ such that $\gamma'(t) = R(\gamma(t))$. Two Reeb orbits are considered the same if they differ only by precomposition with a rotation of \mathbb{R}/T . Given a Reeb orbit $\gamma : \mathbb{R}/T \rightarrow Y$ and a positive integer k , the k -fold **iterate** of γ is the pullback of γ to \mathbb{R}/kT , which we denote by γ^k .

Given a Reeb orbit γ , for any y in the image of γ , the linearization of the Reeb flow along γ defines a symplectic linear map

$$P_{\gamma, y} : (\xi_y, \omega) \longrightarrow (\xi_y, \omega)$$

called the **linearized return map**. The eigenvalues of $P_{\gamma, y}$ do not depend on y . The Reeb orbit γ is said to be **nondegenerate** if $P_{\gamma, y}$ does not have 1 as an eigenvalue. *In this paper we always assume that all Reeb orbits are nondegenerate*⁴. For any Reeb orbit γ , the linearized return map $P_{\gamma, y}$, being symplectic, has eigenvalues λ, λ^{-1} which are either real and positive, in which case γ is called **positive hyperbolic**, or real and negative, in which case γ is called **negative hyperbolic**, or on the unit circle, in which case γ is called **elliptic**.

⁴As K. Cieliebak pointed out to me, it is currently unknown whether an arbitrary stable Hamiltonian structure can be slightly perturbed so as to make all Reeb orbits nondegenerate. (However this is not a problem in the contact case or the mapping torus case.)

2.3 The Conley-Zehnder index

If $\gamma : \mathbb{R}/T \rightarrow Y$ is a Reeb orbit, let $\mathcal{T}(\gamma)$ denote the set of homotopy classes of symplectic trivializations of the 2-plane bundle $\gamma^*\xi$ over $S^1 = \mathbb{R}/T$. This is an affine space over \mathbb{Z} . Our sign convention⁵ is that if $\tau_1, \tau_2 : \gamma^*\xi \rightarrow S^1 \times \mathbb{R}^2$ are two trivializations, then

$$\tau_1 - \tau_2 = \deg(\tau_2 \circ \tau_1^{-1} : S^1 \rightarrow \mathrm{Sp}(2, \mathbb{R}) \approx S^1). \quad (2.1)$$

Now let $\gamma : \mathbb{R}/T \rightarrow Y$ be a Reeb orbit and let τ be a trivialization of $\gamma^*\xi$. Given $t \in \mathbb{R}$, the linearized Reeb flow along γ from time 0 to time t defines a symplectic map $\xi_{\gamma(0)} \rightarrow \xi_{\gamma(t)}$, which with respect to the trivialization τ is a symplectic matrix $\psi(t)$. In particular, $\psi(0)$ is the identity and $\psi(T)$ is the linearized return map. Since γ is assumed nondegenerate, the path of symplectic matrices $\{\psi(t) \mid 0 \leq t \leq T\}$ has a well-defined **Conley-Zehnder index**, which we denote by

$$\mathrm{CZ}_\tau(\gamma) \in \mathbb{Z}.$$

In our three-dimensional situation, this can be described explicitly as follows.

- If γ is hyperbolic, then there is an integer n such that the linearized Reeb flow along γ rotates the eigenspaces of the linearized return map by angle $n\pi$ with respect to τ . In this case

$$\mathrm{CZ}_\tau(\gamma^k) = kn. \quad (2.2)$$

The integer n is even when γ is positive hyperbolic and odd when γ is negative hyperbolic.

- If γ is elliptic, then τ is homotopic to a trivialization in which the linearized Reeb flow along γ rotates by angle $2\pi\theta$. Here the number θ , called the **monodromy angle**, is necessarily irrational because γ and all of its iterates are assumed nondegenerate. In this case

$$\mathrm{CZ}_\tau(\gamma^k) = 2 \lfloor k\theta \rfloor + 1. \quad (2.3)$$

The Conley-Zehnder index depends only on the Reeb orbit γ and the homotopy class of τ in $\mathcal{T}(\gamma)$. If $\tau' \in \mathcal{T}(\gamma)$ is another trivialization, then we have

$$\mathrm{CZ}_\tau(\gamma^k) - \mathrm{CZ}_{\tau'}(\gamma^k) = 2k(\tau' - \tau). \quad (2.4)$$

⁵The paper [11] incorrectly claims to be using this convention. It in fact uses the opposite convention throughout.

2.4 Orbit sets

Definition 2.4. An **orbit set** is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$, where:

- The α_i 's are distinct, embedded Reeb orbits.
- The m_i 's are positive integers⁶.

Define the homology class of α by

$$[\alpha] := \sum_i m_i [\alpha_i] \in H_1(Y).$$

Definition 2.5. If $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ are orbit sets with $[\alpha] = [\beta] \in H_1(Y)$, let $H_2(Y, \alpha, \beta)$ denote the set of relative homology classes of 2-chains Z in Y such that

$$\partial Z = \sum_i m_i \alpha_i - \sum_j n_j \beta_j.$$

That is, two such 2-chains represent the same element of $H_2(Y, \alpha, \beta)$ if and only if their difference is the boundary of a 3-chain. Thus $H_2(Y, \alpha, \beta)$ is an affine space over $H_2(Y)$.

2.5 The relative first Chern class

Fix orbit sets $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ with $[\alpha] = [\beta] \in H_1(Y)$. Also fix trivializations $\tau_i^+ \in \mathcal{T}(\alpha_i)$ for each i and $\tau_j^- \in \mathcal{T}(\beta_j)$ for each j , and denote this set of trivialization choices by τ . Let $Z \in H_2(Y, \alpha, \beta)$.

Definition 2.6. Define the **relative first Chern class**

$$c_\tau(Z) := c_1(\xi|_Z, \tau) \in \mathbb{Z}$$

as follows. Represent Z by a smooth map $f : S \rightarrow Y$, where S is a compact oriented surface with boundary. Choose a section ψ of $f^*\xi$ over S such that ψ is transverse to the zero section, and over each boundary component of S , the section ψ is nonvanishing and has winding number zero with respect to τ . Define

$$c_\tau(Z) := \#\psi^{-1}(0),$$

where ‘ $\#$ ’ denotes the signed count.

⁶Recall that in order to be a generator of the ECH chain complex, α must satisfy the additional requirement that $m_i = 1$ whenever α_i is hyperbolic. However we will not impose that condition anywhere in this paper.

It is not hard to show that $c_\tau(Z)$ is well defined. Moreover

$$c_\tau(Z) - c_\tau(Z') = \langle c_1(\xi), Z - Z' \rangle, \quad (2.5)$$

where $c_1(\xi) \in H^2(Y; \mathbb{Z})$ denotes the ordinary first Chern class. Finally, if $\tau' = (\{\tau_i^{+'}\}, \{\tau_j^{-'}\})$ is another collection of trivialization choices, then

$$c_\tau(Z) - c_{\tau'}(Z) = \sum_i m_i(\tau_i^+ - \tau_i^{+'}) - \sum_j n_j(\tau_j^- - \tau_j^{-'}). \quad (2.6)$$

2.6 Braids around Reeb orbits

Let γ be an embedded Reeb orbit and let m be a positive integer.

Definition 2.7. A **braid** around γ with m strands is an oriented link ζ contained in a tubular neighborhood N of γ such that the tubular neighborhood projection $\zeta \rightarrow \gamma$ is an orientation-preserving degree m submersion.

We now define the writhe, linking number, and winding number of braids around γ , which will be used repeatedly below. For this purpose choose a trivialization τ of $\gamma^*\xi$. Extend the trivialization to identify the tubular neighborhood N with $S^1 \times D^2$, so that the projection of ζ to S^1 is a submersion. Identify $S^1 \times D^2$ with a solid torus in \mathbb{R}^3 via the orientation-preserving diffeomorphism sending

$$(\theta, (x, y)) \longmapsto (1 + x/2)(\cos \theta, \sin \theta, 0) - (0, 0, y/2).$$

This defines an embedding $\phi_\tau : N \rightarrow \mathbb{R}^3$. Now $\phi_\tau(\zeta)$ is an oriented link in \mathbb{R}^3 with no vertical tangents. As such, it has a well-defined writhe, which is the signed count of the crossings in the projection to $\mathbb{R}^2 \times \{0\}$, after perturbing the link to have generic crossings. The sign convention is that counterclockwise twists contribute positively to the writhe.

Definition 2.8. If ζ is a braid around γ , define the **writhe**

$$w_\tau(\zeta) \in \mathbb{Z}$$

to be the writhe of the oriented link $\phi_\tau(\zeta)$ in \mathbb{R}^3 .

This depends only on the isotopy class of ζ and the homotopy class of τ in $\mathcal{T}(\gamma)$. If $\tau' \in \mathcal{T}(\gamma)$ is another trivialization, and if ζ has m strands, then

$$w_\tau(\zeta) - w_{\tau'}(\zeta) = m(m-1)(\tau' - \tau), \quad (2.7)$$

because shifting the trivialization by one adds a full clockwise twist to the braid $\phi_\tau(\zeta)$.

Definition 2.9. If ζ_1 and ζ_2 are disjoint braids around γ , define the **linking number**

$$\ell_\tau(\zeta_1, \zeta_2) \in \mathbb{Z}$$

to be the linking number of the oriented links $\phi_\tau(\zeta_1)$ and $\phi_\tau(\zeta_2)$ in \mathbb{R}^3 . The latter is, by definition, one half the signed count of crossings of a strand of $\phi_\tau(\zeta_1)$ with a strand of $\phi_\tau(\zeta_2)$ in the projection to $\mathbb{R}^2 \times \{0\}$.

Similarly to (2.7), if ζ_k has m_k strands, then

$$\ell_\tau(\zeta_1, \zeta_2) - \ell_{\tau'}(\zeta_1, \zeta_2) = m_1 m_2 (\tau' - \tau). \quad (2.8)$$

Also note that

$$w_\tau(\zeta_1 \cup \zeta_2) = w_\tau(\zeta_1) + w_\tau(\zeta_2) + 2\ell_\tau(\zeta_1, \zeta_2). \quad (2.9)$$

Definition 2.10. If ζ is a braid around γ which is disjoint from γ , define the **winding number**

$$\eta_\tau(\zeta) := \ell_\tau(\zeta, \gamma) \in \mathbb{Z}.$$

2.7 The relative intersection pairing

Definition 2.11. Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ be orbit sets with $[\alpha] = [\beta]$, and let $Z \in H_2(Y, \alpha, \beta)$. An **admissible representative** of Z is a smooth map $f : S \rightarrow [-1, 1] \times Y$, where S is a compact oriented surface with boundary, such that:

- The restriction of f to ∂S consists of positively oriented covers of $\{1\} \times \alpha_i$ with total multiplicity m_i and negatively oriented covers of $\{-1\} \times \beta_j$ with total multiplicity n_j .
- The composition of f with the projection $[-1, 1] \times Y \rightarrow Y$ represents the class Z .
- The restriction of f to the interior of S is an embedding, and f is transverse to $\{-1, 1\} \times Y$.

We will generally abuse notation and denote the admissible representative by S . It is not hard to see that any class Z has an admissible representative; we will construct some special admissible representatives in §3.5 below.

If S is an admissible representative of Z , then for $\epsilon > 0$ sufficiently small, $S \cap (\{1 - \epsilon\} \times Y)$ consists of braids ζ_i^+ with m_i strands in disjoint tubular neighborhoods of the Reeb orbits α_i , which are well defined up to isotopy. Likewise $S \cap (\{-1 + \epsilon\} \times Y)$ consists of disjoint braids ζ_j^- with n_j strands in disjoint tubular neighborhoods of the Reeb orbits β_j . If S' is an admissible

representative of $Z' \in H_2(Y, \alpha', \beta')$, such that the interior of S' does not intersect the interior of S near the boundary, with braids $\zeta_i^{+'}$ and $\zeta_j^{-'}$, define the linking number

$$\ell_\tau(S, S') := \sum_i \ell_\tau(\zeta_i^+, \zeta_i^{+'}) - \sum_j \ell_\tau(\zeta_j^-, \zeta_j^{-'}).$$

Here we are using the same index i for the orbit sets α and α' , so that sometimes $m_i = 0$ or $m'_i = 0$, and likewise the same index j for the orbit sets β and β' ; and τ is a trivialization of ξ over all Reeb orbits in α , α' , β , and β' .

Definition 2.12. If $Z \in H_2(Y, \alpha, \beta)$ and $Z' \in H_2(Y, \alpha', \beta')$, define the **relative intersection number**

$$Q_\tau(Z, Z') \in \mathbb{Z}$$

as follows. Choose admissible representatives S of Z and S' of Z' whose interiors \dot{S} and \dot{S}' are transverse and do not intersect near the boundary. Define

$$Q_\tau(Z, Z') := \#(\dot{S} \cap \dot{S}') - \ell_\tau(S, S').$$

It follows from [11, Lemmas 2.5 and 8.5] that this is well defined, and moreover

$$Q_\tau(Z_1, Z') - Q_\tau(Z_2, Z') = (Z_1 - Z_2) \cdot [\alpha'], \quad (2.10)$$

where ‘ \cdot ’ denotes the ordinary intersection number in Y . Clearly Q_τ is symmetric: $Q_\tau(Z, Z') = Q_\tau(Z', Z)$. Also, it follows from (2.8) that if $\tau' = (\{\tau_i^{+'}\}, \{\tau_j^{-'}\})$ is another collection of trivialization choices, then

$$Q_\tau(Z, Z') - Q_{\tau'}(Z, Z') = \sum_i m_i m'_i (\tau_i^+ - \tau_i^{+'}) - \sum_j n_j n'_j (\tau_j^- - \tau_j^{-'}). \quad (2.11)$$

The most important case is where $Z = Z'$; we denote this by

$$Q_\tau(Z) := Q_\tau(Z, Z).$$

2.8 Definition of the ECH index

Notation 2.13. If $\alpha = \{(\alpha_i, m_i)\}$ is an orbit set and $\tau = \{\tau_i\}$ is a trivialization of ξ over the α_i 's, define

$$\mu_\tau(\alpha) := \sum_i \sum_{k=1}^{m_i} \text{CZ}_{\tau_i}(\alpha_i^k).$$

By equation (2.4), if $\tau' = \{\tau'_i\}$ is another set of trivialization choices, then

$$\mu_\tau(\alpha) - \mu_{\tau'}(\alpha) = \sum_i (m_i^2 + m_i)(\tau'_i - \tau_i). \quad (2.12)$$

Definition 2.14. Let α and β be orbit sets with $[\alpha] = [\beta] = [\Gamma] \in H_1(Y)$, and let $Z \in H_2(Y, \alpha, \beta)$. Define the **ECH index**

$$I(\alpha, \beta, Z) := c_\tau(Z) + Q_\tau(Z) + \mu_\tau(\alpha) - \mu_\tau(\beta).$$

Here τ is a trivialization of ξ over the α_i 's and β_j 's. It follows from equations (2.6), (2.11), and (2.12) that I does not depend on τ .

We can also define a version of I does not depend on a class Z . Namely, by equations (2.5) and (2.10), if $Z' \in H_1(Y, \alpha, \beta)$, then we have the *index ambiguity formula*

$$I(\alpha, \beta, Z) - I(\alpha, \beta, Z') = \langle c_1(\xi) + 2 \text{PD}(\Gamma), Z - Z' \rangle.$$

Here $\text{PD}(\Gamma) \in H^2(Y; \mathbb{Z})$ denotes the Poincare dual of Γ . Thus the following definition makes sense:

Definition 2.15. If α and β are orbit sets with $[\alpha] = [\beta] = \Gamma$, define

$$I(\alpha, \beta) := I(\alpha, \beta, Z) \in \mathbb{Z}/d(c_1(\xi) + 2 \text{PD}(\Gamma)) \quad (2.13)$$

where Z is any class in $H_2(Y, \alpha, \beta)$, and d denotes divisibility in $H^2(Y; \mathbb{Z})$ modulo torsion.

Remark 2.16. It is easy to show, see [11], that I is *additive* in the following sense: if γ is another orbit set with $[\gamma] = \Gamma$, and if $W \in H_2(Y, \beta, \gamma)$, then

$$I(\alpha, \beta, Z) + I(\beta, \gamma, W) = I(\alpha, \gamma, Z + W).$$

Thus I defines a relative grading on ECH generators.

3 An absolute ECH index

We now explain how to refine the relative index $I(\alpha, \beta)$ in (2.13) to an absolute index, which associates to each orbit set a homotopy class of oriented 2-plane fields on Y .

3.1 Homotopy classes of oriented 2-plane fields

Before stating the result, we briefly recall some basic facts about homotopy classes of oriented 2-plane fields which we will need. For proofs of the less obvious of these facts see e.g. [9, §4] and [16, Ch. 28].

Let Y be a connected, closed oriented 3-manifold. Let $\mathcal{P}(Y)$ denote the set of homotopy classes of oriented 2-plane fields on Y . This is the same as the set of homotopy classes of nonvanishing vector fields on Y . Let $\text{Spin}^c(Y)$

denote the set of spin-c structures on Y ; this is an affine space over $H^2(Y; \mathbb{Z})$. There is a surjection

$$\mathfrak{s} : \mathcal{P}(Y) \longrightarrow \text{Spin}^c(Y).$$

Given two oriented 2-plane fields ξ_1 and ξ_2 , the primary obstruction to finding a homotopy between them is the difference between the corresponding spin-c structures,

$$\mathfrak{s}(\xi_1) - \mathfrak{s}(\xi_2) \in H^2(Y; \mathbb{Z}). \quad (3.1)$$

Note that

$$c_1(\xi_1) - c_1(\xi_2) = 2(\mathfrak{s}(\xi_1) - \mathfrak{s}(\xi_2)).$$

In particular, if ξ_1 and ξ_2 determine the same spin-c structure, then $c_1(\xi_1) = c_1(\xi_2)$, and the secondary obstruction to finding a homotopy between them is a class

$$[\xi_1] - [\xi_2] \in \mathbb{Z}/d(c_1(\xi_1)). \quad (3.2)$$

Thus the set of homotopy classes of 2-plane fields determining this spin-c structure is an affine space over $\mathbb{Z}/d(c_1(\xi_1))$. Our sign convention for the affine structure is specified by the isomorphism $\pi_3(S^2) \simeq \mathbb{Z}$ that identifies the Hopf fibration with $+1$.

It will be useful below to understand the obstructions (3.1) and (3.2) in terms of Thom-Pontrjagin theory as follows. Let $\mathcal{L}(Y)$ denote the set of oriented framed links in Y , modulo framed link cobordism. We have a surjection $\mathcal{L}(Y) \rightarrow H_1(Y)$ sending a link L to its homology class $[L] \in H_1(Y)$. There is also a \mathbb{Z} -action on $\mathcal{L}(Y)$ by twisting the framings; on the set of elements of $\mathcal{L}(Y)$ with homology class $\Gamma \in H_1(Y)$, the stabilizer of this \mathbb{Z} action is $2d(\Gamma)$. Our sign convention for this \mathbb{Z} -action is given as in (2.1), but with $\text{Sp}(2, \mathbb{R})$ replaced by $\text{GL}^+(2, \mathbb{R})$. Now fix a trivialization ρ of TY . Then an oriented 2-plane field, regarded as a nonvanishing vector field, defines a map $Y \rightarrow S^2$, and the inverse image of a regular value of this map is an oriented framed link. This construction defines a bijection

$$L_\rho : \mathcal{P}(Y) \longrightarrow \mathcal{L}(Y)$$

satisfying $2[L_\rho(\xi)] = \text{PD}(c_1(\xi))$. In terms of this correspondence, the two-dimensional obstruction (3.1) is given by

$$\mathfrak{s}(\xi_1) - \mathfrak{s}(\xi_2) = \text{PD}([L_\rho(\xi_1)] - [L_\rho(\xi_2)]). \quad (3.3)$$

The three-dimensional obstruction (3.2) is described as follows: if ξ_1 and ξ_2 determine the same spin-c structure, then

$$[\xi_1] - [\xi_2] = L_\rho(\xi_1) - L_\rho(\xi_2). \quad (3.4)$$

This last equation means that the framed link cobordism classes $L_\rho(\xi_1)$ and $L_\rho(\xi_2)$ can be represented by the same link, but with the framings differing by $[\xi_1] - [\xi_2]$.

Now suppose we allow our compact connected oriented 3-manifold Y to have boundary. Let ξ_0 be an oriented rank 2 subbundle of $TY|_{\partial Y}$. Define $\mathcal{P}(Y, \xi_0)$ to be the set of homotopy classes of oriented 2-plane fields on Y that restrict to ξ_0 on ∂Y . (These correspond to spin-c structures \mathfrak{s} on Y together with an isomorphism $\mathfrak{s}|_{\partial Y} \simeq \mathfrak{s}_0$ where \mathfrak{s}_0 is a fixed spin-c structure on ∂Y determined by ξ_0 .) Given two elements $\xi_1, \xi_2 \in \mathcal{P}(Y, \xi_0)$, the primary obstruction to finding a homotopy between them is an element of $H^2(Y, \partial Y; \mathbb{Z})$. The image of this obstruction in $H^2(Y; \mathbb{Z})$, multiplied by 2, equals $c_1(\xi_1) - c_1(\xi_2)$. If the primary obstruction vanishes, then the secondary obstruction is an element of $\mathbb{Z}/d(c_1(\xi_1))$. To describe these obstructions in terms of Thom-Pontrjagin theory, choose a trivialization ρ of TY . This gives rise to a zero-dimensional nullhomologous oriented framed submanifold $F \subset \partial Y$. Let $\mathcal{L}(Y, F)$ denote the set of oriented framed links on Y with boundary F , modulo framed cobordism relative to F . Then as before we have a bijection

$$L_\rho : \mathcal{P}(Y, \xi_0) \longrightarrow \mathcal{L}(Y, F).$$

Given $\xi_1, \xi_2 \in \mathcal{P}(Y, \xi_0)$, the primary obstruction to finding a homotopy between them is Poincaré dual to the difference in relative homology classes $[L_\rho(\xi_1)] - [L_\rho(\xi_2)] \in H_1(Y)$ as in (3.3), and if this vanishes then the secondary obstruction is the difference in framings as in (3.4).

3.2 Statement of the result

Fix a connected closed oriented 3-manifold Y with a stable Hamiltonian structure such that all Reeb orbits are nondegenerate.

Theorem 3.1. *For each orbit set $\alpha = \{(\alpha_i, m_i)\}$, there is a homotopy class of oriented 2-plane fields $I(\alpha) \in \mathcal{P}(Y)$, such that:*

- (a) $I(\alpha)$ is obtained by modifying ξ in a canonical manner (up to homotopy, depending only on m_i) in disjoint tubular neighborhoods of each α_i .
- (b) $\mathfrak{s}(I(\alpha)) = \mathfrak{s}(\xi) + \text{PD}([\alpha])$.
- (c) If α and β are orbit sets with $[\alpha] = [\beta] = \Gamma$, then

$$I(\alpha, \beta) = I(\alpha) - I(\beta) \tag{3.5}$$

in $\mathbb{Z}/d(c_1(\xi) + 2\text{PD}(\Gamma))$.

Remark 3.2. Part (c) asserts that I defines an absolute grading on ECH which refines the relative grading in (2.13). Part (a) implies that the absolute grading I of the empty set⁷ is the homotopy class of the 2-plane field ξ itself. Part (b) asserts that $I(\alpha)$ determines the correct spin-c structure, so that it makes sense to conjecture that Taubes's isomorphism (1.3) between ECH and Seiberg-Witten Floer homology respects the absolute gradings.

3.3 Modifying the 2-plane field near transversal links

To prepare for the proof of the theorem, consider a transversal link $L \subset Y$. This means that L is transverse to the 2-plane field ξ at every point. As such, L has a canonical orientation. Now let τ be a framing of L , i.e. a homotopy class of symplectic trivialization of $\xi|_L$.

Definition 3.3. Given a transversal link L with framing τ , define a homotopy class of oriented 2-plane fields $P_\tau(L)$ as follows.

Let N be a tubular neighborhood of L . On $Y \setminus N$, take $P_\tau(L) := \xi$.

To describe $P := P_\tau(L)$ on N , for each component K of L , let N_K denote the corresponding component of N . Choose a diffeomorphism

$$\phi_K : N_K \xrightarrow{\simeq} S^1 \times D^2$$

such that ϕ_K sends K to $S^1 \times \{0\}$, and the derivative $d\phi_K$ sends $\xi|_K$ to $\{0\} \oplus \mathbb{R}^2$, compatibly with the framing τ . Extend the latter to a trivialization of TN_K , identifying $\xi = \{0\} \oplus \mathbb{R}^2$ and $R = (1, 0, 0)$ at each point. On N_K , choose P , regarded as a vector field, so that:

- On $S^1 \times \{z \in D^2 \mid |z| > 1/2\}$, the vector field P intersects ξ positively.
- On $S^1 \times \{z \in D^2 \mid |z| < 1/2\}$ the vector field P intersects ξ negatively.
- On $S^1 \times \{z \in D^2 \mid |z| = 1/2\}$, the vector field P , regarded using the above trivialization as a function with values in $\mathbb{R} \oplus \mathbb{R}^2$, is given by

$$P(t, e^{i\theta}/2) := (0, e^{-i\theta}). \tag{3.6}$$

These conditions uniquely determine $P_\tau(L)$ up to homotopy.

The following are some basic properties of $P_\tau(L)$.

Lemma 3.4. (a) $\mathfrak{s}(P_\tau(L)) = \mathfrak{s}(\xi) + \text{PD}([L])$.

⁷If Y is a contact manifold, then the empty set is a very important ECH generator, which is a cycle in the ECH chain complex, whose homology class in ECH conjecturally agrees with the contact invariants in the Seiberg-Witten and Heegaard Floer homologies.

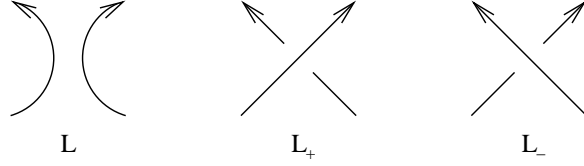


Figure 1: The links L , L_+ , and L_- in Lemma 3.4(c).

(b) If τ' is a different framing of L , then

$$P_\tau(L) - P_{\tau'}(L) \equiv 2(\tau - \tau') \pmod{d(c_1(\xi) + 2\text{PD}([L]))}.$$

(c) If L_\pm is obtained from L by locally fusing two strands into a crossing of sign ± 1 as shown in Figure 1, and if the framings τ of L_\pm and L are the blackboard framing in Figure 1 and agree everywhere else, then

$$P_\tau(L_\pm) - P_\tau(L) \equiv \pm 1 \pmod{d(c_1(\xi) + 2\text{PD}([L]))}.$$

(d) Let $S \subset Y$ be an embedded compact oriented surface with $\partial S = \widehat{L}_1 \sqcup -\widehat{L}_2$ where \widehat{L}_1 and \widehat{L}_2 are transversal links. Let L_0 be a transversal link disjoint from S , let $L_1 := \widehat{L}_1 \sqcup L_0$, and $L_2 := \widehat{L}_2 \sqcup L_0$. Then

$$P_\tau(L_1) - P_\tau(L_2) \equiv c_1(\xi|_S, \tau) \pmod{d(c_1(\xi) + 2\text{PD}([L_1]))},$$

where the framings τ are induced from the conormal direction to S on \widehat{L}_1 and \widehat{L}_2 and an arbitrary framing on L_0 .

Proof. We will prove all four assertions using the Thom-Pontrjagin theory from §3.1.

To start, in the definition of $P_\tau(L)$, we can then take the vector field P on N_K , regarded as a function $S^1 \times D^2 \rightarrow \mathbb{R} \oplus \mathbb{R}^2$, to be

$$P(t, re^{i\theta}) = (-\cos(\pi r), \sin(\pi r)e^{-i\theta}). \quad (3.7)$$

Then $(-1, 0)$ is a regular value of P , whose inverse image is the core circle $S^1 \times \{0\} \subset S^1 \times D^2$. This circle is oriented positively. In terms of the Thom-Pontrjagin construction, this means that on N_K ,

$$[L_\rho(P)] - [L_\rho(\xi)] = [K] \in H_1(N_K).$$

Together with (3.3), this implies assertion (a).

To calculate the framing on $L_\rho(P)$ above, we can take a nearby regular value of P in S^2 such as $(0, 1, 0)$. The inverse image of this is $S^1 \times \{(1/2, 0)\}$,

and so $L_\rho(P)$ has framing 0 with respect to our trivializations. Now let τ' be another framing of L such that on K we have $\tau - \tau' = k \in \mathbb{Z}$. The difference in trivializations

$$\Phi := \tau' \circ \tau^{-1} : S^1 \longrightarrow \mathrm{Sp}(2, \mathbb{R})$$

can be taken to be

$$\Phi(t) = e^{ikt}.$$

This is induced by a diffeomorphism $\tilde{\Phi} : S^1 \times D^2 \rightarrow S^1 \times D^2$ given by

$$\tilde{\Phi}(t, re^{i\theta}) = (t, re^{i(\theta+kt)}).$$

With respect to the previous trivialization ρ of TN_K coming from τ , a vector field P' corresponding to τ' is given by the function

$$P' = (1 \oplus \Phi^{-1}) \circ P \circ \tilde{\Phi},$$

where P is the function defined by (3.7). This comes out to be

$$P'(t, re^{i\theta}) = (-\cos(\pi r), \sin(\pi r)e^{-i(\theta+2kt)}).$$

Again, the link $L_\rho(P')$ on $S^1 \times S^2$ is the circle $S^1 \times \{0\}$, oriented positively. However now the regular value $(0, 1, 0)$ of the map P' to S^2 has inverse image $\{(t, \frac{1}{2}e^{-2ikt})\}$, so $L_\rho(P')$ has framing $-2k$. Together with (3.4), this implies assertion (b) of the lemma.

Assertion (c) follows immediately from the Thom-Pontrjagin construction.

We now prove assertion (d). To start, we may assume that S has no closed components. For if S_0 is the union of the closed components of S , then

$$c_1(\xi|_{S_0}, \tau) = \langle c_1(\xi), [S_0] \rangle = \langle c_1(\xi) + 2\mathrm{PD}([L_1]), [S_0] \rangle,$$

since S_0 is disjoint from L_1 . Thus removing S_0 from S does not affect the validity of the congruence that we need to prove.

We may then also assume that S is connected, since tubing together different components of S has no effect on $c_1(\xi|_S, \tau)$.

Now let $N \subset Y$ be a neighborhood of S , identified with $\tilde{S} \times [-1, 1]$, where $\tilde{S} \supset S$ is obtained by extending S slightly past its boundary, so that the identification $N \simeq \tilde{S} \times [-1, 1]$ sends S to $S \times \{0\}$. Choose N small enough so that it is disjoint from L_0 . It is enough to show that in $\mathcal{P}(N, \xi|_{\partial N})$ we have

$$P_\tau(\hat{L}_1) - P_\tau(\hat{L}_2) = c_1(\xi|_S, \tau) \in \mathbb{Z}. \quad (3.8)$$

Since S has nonempty boundary, we can choose a trivialization $\rho : TN \rightarrow N \times \mathbb{R}^3$ identifying TS with $S \times (\mathbb{R}^2 \oplus \{0\})$. We can choose this trivialization so that $(0, 0, \pm 1)$ is a regular value of the map $S \rightarrow S^2$ given by the normalized

Reeb vector field; let $T_{\pm} \subset S$ denote the inverse image of $(0, 0, \pm 1)$ under this map, and let $t_{\pm} \in \mathbb{Z}$ denote the signed count of points in the set T_{\pm} .

For $i = 1, 2$, we can take $L_{\rho}(P_{\tau}(\widehat{L}_i))$ to be the set of points in N such that the vector field corresponding to $P_{\tau}(\widehat{L}_i)$ points in the direction $(0, 0, 1)$. If this vector field is chosen appropriately, then the framed link $L_{\rho}(P_{\tau}(\widehat{L}_i))$ consists of the following:

- A vertical line segment at each point in T_+ , such that the number of upward pointing segments minus the number of downward pointing segments equals t_+ .
- A vertical pushoff of the link \widehat{L}_i , with the conormal framing.

It follows using assertion (c) and equation (3.4) that

$$P_{\tau}(\widehat{L}_1) - P_{\tau}(\widehat{L}_2) = 2t_+ - \chi(S).$$

Applying the Poincare-Hopf index theorem to the projection of R onto TS gives

$$t_+ - t_- = \chi(S).$$

On the other hand, the projection of $(0, 0, 1)$ to ξ defines a section of $\xi|_S$ which is zero exactly where the Reeb vector field is vertical, showing that

$$t_+ + t_- = c_1(\xi|_S, \tau),$$

compare [7, §4.2]. Combining the above three equations proves (3.8). \square

Remark 3.5. One can define another homotopy class of oriented 2-plane fields $P'(L)$, following Definition 3.3, but with equation (3.6) replaced by

$$P'(t, e^{i\theta}/2) := (0, e^{i\theta}).$$

The homotopy class of oriented 2-plane fields $P'(L)$ satisfies $\mathfrak{s}(P'(L)) = \mathfrak{s}(\xi) - \text{PD}([L])$, does not depend on a framing of L , satisfies the analogue of property (c) above, and the analogue of property (d) but with the opposite sign. When ξ is a contact structure, the homotopy class $P'(L)$ corresponds to the contact structure obtained from ξ by a *Lutz twist* along L , see e.g. [8]. Although $P'(L)$ is not relevant for Theorem 3.1, it is significant in connection with defining a relative filtration on ECH, see Proposition 6.5.

3.4 Definition of the absolute grading

Definition 3.6. Given an orbit set $\alpha = \{(\alpha_i, m_i)\}$, define a homotopy class of oriented 2-plane fields $I(\alpha) \in \mathcal{P}(Y)$ as follows. Choose trivializations $\tau = \{\tau_i\}$ of ξ over the α_i 's. For each i , choose a braid ζ_i around α_i with m_i

strands. Assume that the ζ_i 's are in disjoint tubular neighborhoods of the α_i 's. Consider the transverse link $L := \bigcup_i \zeta_i$, with the framing τ induced by the τ_i 's. Define

$$I(\alpha) := P_\tau(L) - \sum_i w_{\tau_i}(\zeta_i) + \mu_\tau(\alpha). \quad (3.9)$$

Lemma 3.7. *$I(\alpha)$ is well-defined.*

Proof. First fix the trivialization choices and consider replacing the braids ζ_i by some other braids ζ'_i . Then by Lemma 3.4(c), we have

$$P_\tau(L) - P_\tau(L') = \sum_i (w_{\tau_i}(\zeta_i) - w_{\tau_i}(\zeta'_i)).$$

Thus for given trivializations, $I(\alpha)$ does not depend on the choice of braids.

Now fix the braids and consider a different set of trivialization choices $\tau' = \{\tau'_i\}$. Changing the trivialization over α_i from τ'_i to τ_i shifts the induced framing on ζ_i by $m_i(\tau_i - \tau'_i)$. Thus by Lemma 3.4,

$$P_\tau(L) - P_{\tau'}(L) = \sum_i 2m_i(\tau_i - \tau'_i).$$

Combining this with equations (2.7) and (2.12) proves that $I(\alpha)$ does not depend on the trivialization choices. \square

We now want to prove that $I(\alpha)$ satisfies properties (a), (b), and (c) in Theorem 3.1. Property (a) is clear from the proof of Lemma 3.7. Property (b) is immediate from Lemma 3.4(a).

3.5 Computing Q using embedded surfaces in Y

To prepare for the proof of Theorem 3.1(c), we now establish a general formula for the relative intersection pairing Q in terms of embedded surfaces in Y . We use the notation from §2.7.

Definition 3.8. An admissible representative S of a class $Z \in H_2(Y, \alpha, \beta)$ is **nice** if the projection of S to Y is an immersion, and the projection of the interior \dot{S} to Y is an embedding which does not intersect the α_i 's or β_j 's.

Lemma 3.9. *If none of the α_i 's equals any of the β_j 's, then every class $Z \in H_2(Y, \alpha, \beta)$ has a nice representative.*

Proof. Let N be the union of disjoint tubular neighborhoods of the α_i 's and β_j 's. Then Z determines a relative homology class in

$$H_2(Y \setminus N, \partial N) = H^1(Y \setminus N; \mathbb{Z}) = [Y \setminus N, S^1].$$

The latter can be represented by an embedded oriented surface $S_0 \subset Y \setminus N$ transverse to ∂N . On each component of ∂N , one can successively cap off contractible circles in $S_0 \cap \partial N$, then cancel adjacent parallel arcs with opposite orientations, then straighten the remaining arcs, to arrange that $S_0 \cap \partial N$ is a union of torus braids around each α_i with m_i strands intersecting ξ positively, and around each β_j with n_j strands intersecting ξ negatively. (These braids have the correct number of strands because of our assumption that none of the α_i 's equals any of the β_j 's.) We can now fill in S_0 over N and lift it to $\mathbb{R} \times Y$ to obtain the desired nice representative. \square

If S is a nice representative of Z with associated braids ζ_i^+ and ζ_j^- , then it makes sense to define the winding number

$$\eta_\tau(S) := \sum_i \eta_{\tau_i^+}(\zeta_i^+) - \sum_j \eta_{\tau_j^-}(\zeta_j^-).$$

Also, if S is any admissible representative of Z , define the writhe

$$w_\tau(S) := \sum_i w_{\tau_i^+}(\zeta_i^+) - \sum_j w_{\tau_j^-}(\zeta_j^-). \quad (3.10)$$

Lemma 3.10. *Suppose that S is a nice representative of Z . Then*

$$Q_\tau(Z) = -w_\tau(S) - \eta_\tau(S).$$

Proof. Choose a smooth function $\varphi : [-1, 1] \rightarrow [-1, 1]$ such that $\varphi(s) \geq s$, with equality only for $s \in \{\pm 1\}$. Make another admissible representative S' of Z by composing S with the diffeomorphism from $[-1, 1] \times Y$ to itself that sends $(s, y) \mapsto (\varphi(s), y)$. The corresponding braid $\zeta_i^{+'}$ is obtained by pushing ζ_i^+ radially towards α_i , so their linking number is given by

$$\ell_{\tau_i^+}(\zeta_i^+, \zeta_i^{+'}) = w_{\tau_i^+}(\zeta_i^+) + \eta_{\tau_i^+}(\zeta_i^+).$$

Combining this with an analogous formula for the negative braids, we obtain

$$\ell_\tau(S, S') = w_\tau(S) + \eta_\tau(S).$$

On the other hand, since the projection of S to Y is an embedding on the interior, it follows that S does not intersect S' on the interior, so

$$\#(\dot{S} \cap \dot{S}') = 0.$$

The lemma now follows from the definition of Q . \square

If α and β have some Reeb orbits in common, then a nice representative of Z might not exist, but the above formula for $Q_\tau(Z)$ can be extended to this case as follows.

To start, it follows from the definition that Q_τ is quadratic in the following sense: if $Z \in H_2(Y, \alpha, \beta)$ and $Z' \in H_2(Y, \alpha', \beta')$, then

$$Z + Z' \in H_2(Y, \alpha\alpha', \beta\beta')$$

is defined (here the product of two orbit sets is defined by adding the multiplicities of all Reeb orbits involved), and

$$Q_\tau(Z + Z') = Q_\tau(Z) + 2Q_\tau(Z, Z') + Q_\tau(Z'). \quad (3.11)$$

Here τ is a trivialization of ξ over all Reeb orbits under consideration.

Now let $\widehat{\alpha}$ and $\widehat{\beta}$ be obtained from α and β by “dividing by their greatest common factor” according to the following procedure: Whenever $\alpha_i = \beta_j$, replace m_i by $\widehat{m}_i := m_i - \min(m_i, n_j)$ and replace n_j by $\widehat{n}_j := n_j - \min(m_i, n_j)$; then discard all pairs $(\alpha_i, \widehat{m}_i)$ with $\widehat{m}_i = 0$ and (β_j, \widehat{n}_j) with $\widehat{n}_j = 0$. Now $\widehat{\alpha}$ and $\widehat{\beta}$ have no Reeb orbits in common. Let γ denote the “greatest common factor” of α and β , namely

$$\gamma := \{(\alpha_i, m_i - \widehat{m}_i) \mid m_i > \widehat{m}_i\} = \{(\beta_j, n_j - \widehat{n}_j) \mid n_j > \widehat{n}_j\},$$

so that $\alpha = \widehat{\alpha}\gamma$ and $\beta = \widehat{\beta}\gamma$.

Any class $Z \in H_2(Y, \alpha, \beta)$ can be uniquely written as $Z = Z_0 + \widehat{Z}$, where $Z_0 \in H_2(Y, \gamma, \gamma)$ corresponds to 0 under the obvious identification $H_2(Y, \gamma, \gamma) = H_2(Y)$, and $\widehat{Z} \in H_2(Y, \widehat{\alpha}, \widehat{\beta})$. And \widehat{Z} has a nice representative by Lemma 3.9.

Lemma 3.11. *Given $Z \in H_2(Y, \alpha, \beta)$, let \widehat{S} be a nice representative of the corresponding class $\widehat{Z} \in H_2(Y, \widehat{\alpha}, \widehat{\beta})$, with associated braids $\widehat{\zeta}_i^+$ and $\widehat{\zeta}_j^-$. Choose the trivializations so that $\tau_i^+ = \tau_j^-$ whenever $\alpha_i = \beta_j$. Define*

$$\ell_\tau(\widehat{S}, \mathbb{R} \times \gamma) := \sum_i (m_i - \widehat{m}_i) \eta_{\tau_i^+}(\widehat{\zeta}_i^+) - \sum_j (n_j - \widehat{n}_j) \eta_{\tau_j^-}(\widehat{\zeta}_j^-).$$

Then

$$Q_\tau(Z) = -w_\tau(\widehat{S}) - \eta_\tau(\widehat{S}) - 2\ell_\tau(\widehat{S}, \mathbb{R} \times \gamma).$$

Proof. It follows easily from the definition of Q , and our assumption that $\tau_i^+ = \tau_j^-$ whenever $\alpha_i = \beta_j$, that $Q_\tau(Z_0) = 0$. So by equation (3.11) and Lemma 3.10, it is enough to show that

$$Q_\tau(\widehat{Z}, Z_0) = -\ell_\tau(\widehat{S}, \mathbb{R} \times \gamma). \quad (3.12)$$

We can find an admissible representative S_0 of the class Z_0 which is contained in a union of disjoint tubular neighborhoods of the cylinders $\mathbb{R} \times \alpha_i$ for those Reeb orbits α_i that equal some β_j . Since the interior of \widehat{S} does not intersect the α_i 's or β_j 's, we can arrange for the interior of S_0 to be disjoint from the interior of \widehat{S} , so that

$$\#(\widehat{S}, \dot{S}_0) = 0.$$

At the same time we can arrange that the braids associated to S_0 are contained in tubular neighborhoods of the α_i 's and β_j 's that do not intersect the braids $\widehat{\zeta}_i^+$ and $\widehat{\zeta}_j^-$, which implies that

$$\ell_\tau(\widehat{S}, S_0) = \ell_\tau(\widehat{S}, \mathbb{R} \times \gamma).$$

Putting the above two equations into the definition of Q proves (3.12). \square

3.6 The absolute grading determines the relative

Proof of Theorem 3.1(c). Pick an arbitrary class $Z \in H_2(Y, \alpha, \beta)$. Choose trivializations $\tau_+ = \{\tau_i^+\}$ of ξ over the α_i 's and trivializations $\tau_- = \{\tau_j^-\}$ of ξ over the β_j 's. By the definitions of the relative and absolute versions of I , to prove the desired identity (3.5), we need to show that

$$P_{\tau_+}(L_+) - P_{\tau_-}(L_-) - \sum_i w_{\tau_i^+}(\zeta_i^+) + \sum_j w_{\tau_j^-}(\zeta_j^-) \equiv c_\tau(Z) + Q_\tau(Z) \quad (3.13)$$

modulo $d(c_1(\xi) + 2\text{PD}([\Gamma]))$, where ζ_i^+ is some braid around α_i with m_i strands, and ζ_j^- is some braid around β_j with n_j strands, and $L_+ := \sqcup_i \zeta_i^+$ and $L_- := \sqcup_j \zeta_j^-$.

We will prove (3.13) for special braids ζ_i^+ and ζ_j^- chosen as follows. First define $\widehat{\alpha}$ and $\widehat{\beta}$ by “dividing α and β by their greatest common factor” as explained in §3.5. By Lemma 3.9, we can find a nice representative \widehat{S} of the class $\widehat{Z} \in H_2(Y, \widehat{\alpha}, \widehat{\beta})$ determined by Z . Take the projection of \widehat{S} to Y , and remove its intersection with a union of small disjoint tubular neighborhoods of the α_i 's and β_j 's, to obtain an embedded compact oriented surface S in Y , whose boundary is a transverse link. More precisely,

$$\partial S = \bigsqcup_i \widehat{\zeta}_i^+ \sqcup \bigsqcup_j -\widehat{\zeta}_j^-,$$

where $\widehat{\zeta}_i^+$ is a braid around α_i with \widehat{m}_i strands which does not intersect α_i , and $\widehat{\zeta}_j^-$ is a braid around β_j with \widehat{n}_j strands which does not intersect β_j .

Now define ζ_i^+ and ζ_j^- as follows. If α_i does not equal any β_j , take $\zeta_i^+ := \widehat{\zeta}_i^+$; likewise if β_j does not equal any α_i , take $\zeta_j^- := \widehat{\zeta}_j^-$. If $\alpha_i = \beta_j$,

choose an arbitrary braid ζ_{ij} around this Reeb orbit with $m_i - \widehat{m}_i = n_j - \widehat{n}_j$ strands, such that ζ_{ij} is contained in a small tubular neighborhood of $\alpha_i = \beta_j$ which does not intersect $\widehat{\zeta}_i^+$ or $\widehat{\zeta}_j^-$; then take $\zeta_i^+ := \widehat{\zeta}_i^+ \sqcup \zeta_{ij}$ and $\zeta_j^- := \widehat{\zeta}_j^- \sqcup \zeta_{ij}$.

We now prove (3.13) for these choices. By equation (2.9), if $\alpha_i = \beta_j$, then

$$\begin{aligned} w_{\tau_i^+}(\zeta_i^+) &= w_{\tau_i^+}(\widehat{\zeta}_i^+) + w_{\tau_i^+}(\zeta_{i,j}) + 2(m_i - \widehat{m}_i)\eta_{\tau_i^+}(\widehat{\zeta}_i^+), \\ w_{\tau_j^-}(\zeta_j^-) &= w_{\tau_j^-}(\widehat{\zeta}_j^-) + w_{\tau_j^-}(\zeta_{i,j}) + 2(n_j - \widehat{n}_j)\eta_{\tau_j^-}(\widehat{\zeta}_j^-). \end{aligned}$$

Thus if we choose the trivializations so that $\tau_i^+ = \tau_j^-$ whenever $\alpha_i = \beta_j$, then in the notation of Lemma 3.11,

$$\sum_i w_{\tau_i^+}(\zeta_i^+) - \sum_j w_{\tau_j^-}(\zeta_j^-) = w_\tau(\widehat{S}) + 2\ell_\tau(\widehat{S}, \mathbb{R} \times \gamma).$$

Thus by Lemma 3.11, our goal (3.13) is equivalent to

$$P_{\tau_+}(L_+) - P_{\tau_-}(L_-) \equiv c_\tau(Z) - \eta_\tau(\widehat{S}). \quad (3.14)$$

To prove (3.14), let τ^ν denote the framing of L_\pm induced by the conormal direction to S , together with some fixed framings of the braids $\zeta_{i,j}$. Then by Lemma 3.4(d),

$$P_{\tau^\nu}(L_+) - P_{\tau^\nu}(L_-) \equiv c_1(\xi|_S, \tau^\nu).$$

Now on each component C of the braid $\widehat{\zeta}_i^+$, the conormal framing τ^ν differs from the framing induced by τ_i^+ by the winding number $\eta_{\tau_i^+}(C)$. Likewise for the braids $\widehat{\zeta}_j^-$. This framing difference has two consequences. First,

$$c_1(\xi|_S, \tau^\nu) = c_\tau(Z) + \eta_\tau(\widehat{S}).$$

Second, using Lemma 3.4(b),

$$(P_{\tau^\nu}(L_+) - P_{\tau^\nu}(L_-)) - (P_{\tau_+}(L_+) - P_{\tau_-}(L_-)) \equiv 2\eta_\tau(\widehat{S}).$$

Combining the above three equations proves (3.14). \square

4 The index inequality in cobordisms

The main result of this section is Theorem 4.15 below, which generalizes the basic ECH index inequality (1.1) to symplectic cobordisms.

4.1 Cobordism setup

Let $(Y_+, \lambda_+, \omega_+)$ and $(Y_-, \lambda_-, \omega_-)$ be closed oriented 3-manifolds with stable Hamiltonian structures. Write $E_+ := [0, \infty) \times Y_+$ and $E_- := (-\infty, 0] \times Y_-$. Let s denote the $[0, \infty)$ or $(-\infty, 0]$ coordinate on E_\pm .

Definition 4.1. A **symplectic cobordism** from Y_+ to Y_- is a smooth 4-manifold X with a decomposition

$$X = E_- \cup_{Y_-} \overline{X} \cup_{Y_+} E_+,$$

where (\overline{X}, ω) is a compact symplectic 4-manifold such that $\partial\overline{X} = -Y_- \sqcup Y_+$ and $\omega|_{Y_\pm} = \omega_\pm$. In the special case $(Y_\pm, \lambda_\pm, \omega_\pm) = (Y, \lambda, \omega)$, we allow $\overline{X} = \emptyset$, in which case $X = \mathbb{R} \times Y$ is called the **symplectization** of Y .

We will not really use the symplectic form on \overline{X} in the present paper, but it enables compactness results for holomorphic curves [1, 3].

Definition 4.2. Let X be a symplectic cobordism from $(Y_+, \lambda_+, \omega_+)$ to $(Y_-, \lambda_-, \omega_-)$. An almost complex structure J on X is **admissible** if:

- On E_\pm , the almost complex structure J is independent of s , sends ∂_s to R_\pm , and sends ξ_\pm to itself compatibly with ω_\pm .
- On \overline{X} , the almost complex structure J is tamed by ω .

4.2 The ECH index in cobordisms

Fix a symplectic cobordism as above. Suppose $\alpha^+ = \{(\alpha_i^+, m_i^+)\}$ is an orbit set in Y_+ , and $\alpha^- = \{(\alpha_j^-, m_j^-)\}$ is an orbit set in Y_- , such that $[\alpha^+] \in H_1(Y_+)$ and $[\alpha^-] \in H_1(Y_-)$ map to the same homology class in $H_1(X)$. Let $H_2(X, \alpha^+, \alpha^-)$ denote the set of relative homology classes of 2-chains Z in the 4-manifold X with

$$\partial Z = \sum_i m_i^+ \{1\} \times \alpha_i^+ - \sum_j m_j^- \{-1\} \times \alpha_j^-.$$

This is an affine space over $H_2(X)$. Note that in the special case when $X = \mathbb{R} \times Y$, this is canonically isomorphic to the affine space $H_2(Y, \alpha^+, \alpha^-)$ from Definition 2.5, via the projection $\mathbb{R} \times Y \rightarrow Y$.

Returning to the general case, let $Z \in H_2(X, \alpha^+, \alpha^-)$. If τ is a homotopy class of trivialization of ξ_+ over the Reeb orbits α_i^+ and of ξ_- over the Reeb orbits α_j^- , define the relative first Chern class

$$c_\tau(Z) := c_1(TX|_Z, \tau) \in \mathbb{Z},$$

generalizing Definition 2.6, as follows. Regard TX as a complex vector bundle via any admissible almost complex structure. Fix a trivialization

$$TX|_{\{1\} \times \alpha_i^+} \xrightarrow{\cong} \alpha_i^+ \times (\mathbb{C} \oplus \mathbb{C})$$

sending ξ_+ to the first summand via τ and sending ∂_s and R_+ to 1 and $\sqrt{-1}$ respectively in the second summand. Choose an analogous trivialization of TX over $\{-1\} \times \alpha_j^-$. Represent Z by a smooth map $f : S \rightarrow X$ where S is a compact surface with boundary. Choose a generic section ψ of $f^*(\wedge^2 TX)$ which on ∂S is nonvanishing and has winding number zero with respect to the above trivialization. Then define $c_1(TX|_Z, \tau) := \#\psi^{-1}(0)$.

If $Z \in H_2(X, \alpha^+, \alpha^-)$ and $Z' \in H_2(X, \alpha^{\pm'}, \alpha^{-'})$, and if τ is a trivialization of ξ_{\pm} over all orbits in α^{\pm} and $\alpha^{\pm'}$, then $Q_{\tau}(Z, Z') \in \mathbb{Z}$ is defined by obvious analogy with Definition 2.12.

Definition 4.3. If $Z \in H_2(X, \alpha^+, \alpha^-)$, define the ECH index

$$I(Z) := c_{\tau}(Z) + Q_{\tau}(Z) + \mu_{\tau}(\alpha^+) - \mu_{\tau}(\alpha^-).$$

4.3 Holomorphic curves

Fix a symplectic cobordism X with an admissible almost complex structure J . Recall that a **holomorphic curve** in X is a map

$$u : (C, j) \longrightarrow (X, J)$$

where (C, j) is a Riemann surface and $J \circ du = du \circ j$. One declares that $u : (C, j) \rightarrow (X, J)$ is equivalent to $u' : (C', j') \rightarrow (X, J)$ if and only if there is a biholomorphic map $\varphi : (C, j) \rightarrow (C', j')$ such that $u' \circ \varphi = u$.

In this paper we will always assume further that:

- (C, j) is a punctured compact Riemann surface, possibly disconnected.
- u is nonconstant on each component of C .
- Each end of u is either asymptotic to $[0, \infty) \times \gamma$ for some Reeb orbit γ in Y_+ , or asymptotic to $(-\infty, 0] \times \gamma$ for some Reeb orbit γ in Y_- .

If γ is an embedded Reeb orbit in Y_+ and k is a positive integer, a **positive end** of u at γ of multiplicity k is an end of u which is asymptotic to $[0, \infty) \times \gamma^k$. Recall here that γ^k denotes the k -fold iterate of γ . Likewise, if γ is an embedded Reeb orbit in Y_- , a **negative end** of u at γ of multiplicity k is an end of u which is asymptotic to $(-\infty, 0] \times \gamma^k$.

Definition 4.4. A holomorphic curve $u : C \rightarrow X$ is **multiply covered** if there is a subset $C' \subset C$ which is a union of components, a holomorphic branched cover $\varphi : C' \rightarrow C_0$ of degree > 1 , and a holomorphic map $u_0 : C_0 \rightarrow X$ such that $u|_{C'} = u_0 \circ \varphi$. Otherwise u is called **simple**. We say that u is **irreducible** if its domain C is connected.

Let $\alpha^+ = \{(\alpha_i^+, m_i^+)\}$ and $\alpha^- = \{(\alpha_j^-, m_j^-)\}$ be orbit sets in Y_+ and Y_- with the same homology class in X .

Definition 4.5. Let $\mathcal{M}(\alpha^+, \alpha^-)$ denote the moduli space of holomorphic curves u in X with:

- positive ends at α_i^+ with total multiplicity m_i^+ , for each i ;
- negative ends at α_j^- with total multiplicity m_j^- , for each j ;

and no other ends.

Any such u determines a relative homology class $[u] \in H_2(X, \alpha^+, \alpha^-)$, after using orientation-preserving diffeomorphisms $[0, \infty) \simeq [0, 1)$ and $(-\infty, 0] \simeq (-1, 0]$ to identify

$$X \simeq ((-1, 0] \times Y_-) \cup_{Y_-} \overline{X} \cup_{Y_+} ([0, 1) \times Y_+).$$

Definition 4.6. Given $Z \in H_2(X, \alpha^+, \alpha^-)$, let

$$\mathcal{M}(\alpha^+, \alpha^-, Z) := \{u \in \mathcal{M}(\alpha^+, \alpha^-) \mid [u] = Z\}.$$

Notation 4.7. We will often abuse notation and refer to the holomorphic curve $u : (C, j) \rightarrow (X, J)$ simply by C . If τ is a trivialization of ξ over the Reeb orbits α_i^+ and α_j^- , we write $c_\tau(C) := c_\tau([C])$; $Q_\tau(C) := Q_\tau([C])$; $\mu_\tau(C) := \mu_\tau(\alpha^+) - \mu_\tau(\alpha^-)$; and

$$I(C) := I(\alpha^+, \alpha^-, [C]) = c_\tau(C) + Q_\tau(C) + \mu_\tau(C).$$

Example 4.8. If C is closed, representing a homology class $[C] \in H_2(X)$, then

$$I(C) = \langle c_1(TX), [C] \rangle + [C] \cdot [C].$$

Taubes's Gromov invariant [23] of a closed symplectic 4-manifold X counts (in a subtle way) holomorphic curves C in X with $I(C) = 0$.

4.4 The relative adjunction formula

Consider now a *simple* J -holomorphic curve $C \in \mathcal{M}(\alpha^+, \alpha^-, Z)$. It follows from [21, Cor. 2.6] that C is embedded except possibly for finitely many singularities. We then have the following relative adjunction formula:

Proposition 4.9. *If C is a simple holomorphic curve in X as above, then*

$$c_\tau(C) = \chi(C) + Q_\tau(C) + w_\tau(C) - 2\delta(C). \quad (4.1)$$

Here $\delta(C)$ is a count of the singularities of C in X with positive integer weights as in [18, §7]. The weight of a singular point p is the number of self-intersections of a perturbation of C to a generic holomorphic immersion in a neighborhood of p . Also, $w_\tau(C)$ is the **asymptotic writhe** of C , defined by obvious analogy with (3.10). A proof of the relative adjunction formula in a slightly different context can be found in [11, §3], and this carries over in a straightforward manner to the present situation.

Example 4.10. If C is closed, then there is no writhe term or trivialization choice, and (4.1) reduces to the usual adjunction formula

$$\langle c_1(TX), [C] \rangle = \chi(C) + [C] \cdot [C] - 2\delta(C). \quad (4.2)$$

4.5 The Fredholm index

Let $C \in \mathcal{M}(\alpha^+, \alpha^-)$. For each i , let n_i^+ denote the number of positive ends of u at α_i^+ , and let $\{q_{i,k}^+\}_{k=1}^{n_i^+}$ denote their multiplicities. Likewise, for each j , let n_j^- denote the number of negative ends of u at α_j^- , and let $\{q_{j,k}^-\}_{k=1}^{n_j^-}$ denote their multiplicities. Thus $\sum_{k=1}^{n_i^+} q_{i,k}^+ = m_i^+$ and $\sum_{k=1}^{n_j^-} q_{j,k}^- = m_j^-$.

Notation 4.11. If τ is a trivialization of ξ_\pm over the orbits in α^\pm , define

$$\mu_\tau^0(C) := \sum_i \sum_{k=1}^{n_i^+} \text{CZ}_\tau((\alpha_i^+)^{q_{i,k}^+}) - \sum_j \sum_{k=1}^{n_j^-} \text{CZ}_\tau((\alpha_j^-)^{q_{j,k}^-}).$$

That is, $\mu_\tau^0(C)$ is the sum of the Conley-Zehnder indices of the positive ends of C , minus the sum of the CZ indices of the negative ends of C . This should be contrasted with $\mu_\tau(C)$, which is a sum of many more Conley-Zehnder terms:

$$\mu_\tau(C) = \sum_i \sum_{l=1}^{m_i^+} \text{CZ}_\tau((\alpha_i^+)^l) - \sum_j \sum_{l=1}^{m_j^-} \text{CZ}_\tau((\alpha_j^-)^l).$$

Definition 4.12. Define the **Fredholm index**

$$\text{ind}(C) := -\chi(C) + 2c_\tau(C) + \mu_\tau^0(C). \quad (4.3)$$

It is shown in [4], using an index formula from [20], that if J is generic and C is simple, then the moduli space $\mathcal{M}(\alpha^+, \alpha^-, [C])$ is a manifold near C of dimension $\text{ind}(C)$. However in the present paper we do not assume that J is generic.

4.6 Incoming and outgoing partitions

Before stating the index inequality, we need a digression to introduce some special partitions associated to Reeb orbits.

Let Y be a three-manifold with a stable Hamiltonian structure, let γ be an embedded Reeb orbit in Y , and let m be a positive integer.

Definition 4.13. Define two partitions of m , the **incoming partition** $P_\gamma^{\text{in}}(m)$ and the **outgoing partition** $P_\gamma^{\text{out}}(m)$, as follows.

- If γ is positive hyperbolic, then

$$P_\gamma^{\text{in}}(m) := P_\gamma^{\text{out}}(m) := (1, \dots, 1).$$

- If γ is negative hyperbolic, then

$$P_\gamma^{\text{in}}(m) := P_\gamma^{\text{out}}(m) := \begin{cases} (2, \dots, 2), & m \text{ even,} \\ (2, \dots, 2, 1), & m \text{ odd.} \end{cases}$$

- If γ is elliptic with monodromy angle θ , then $P_\gamma^{\text{in}}(m) := P_\theta^{\text{in}}(m)$ and $P_\gamma^{\text{out}}(m) := P_\theta^{\text{out}}(m)$, where $P_\theta^{\text{in}}(m)$ and $P_\theta^{\text{out}}(m)$ are defined below.

Definition 4.14. Let θ be an irrational number and let m be a positive integer. Define partitions $P_\theta^{\text{in}}(m)$ and $P_\theta^{\text{out}}(m)$ of m as follows.

Let $\Lambda_\theta^{\text{in}}(m)$ denote the lowest convex polygonal path in the plane that starts at $(0, 0)$, ends at $(m, \lceil m\theta \rceil)$, stays above the line $y = \theta x$, and has corners at lattice points. Then the integers in $P_\theta^{\text{in}}(m)$ are the horizontal displacements of the segments of the path $\Lambda_\theta^{\text{in}}(m)$ between lattice points.

Likewise, let $\Lambda_\theta^{\text{out}}(m)$ denote the highest concave polygonal path in the plane that starts at $(0, 0)$, ends at $(m, \lfloor m\theta \rfloor)$, stays below the line $y = \theta x$, and has corners at lattice points. Then the integers in $P_\theta^{\text{out}}(m)$ are the horizontal displacements of the segments of the path $\Lambda_\theta^{\text{out}}(m)$ between lattice points.

Note that $P_\theta^{\text{in}}(m)$ and $P_\theta^{\text{out}}(m)$ depend only on the class of θ in \mathbb{R}/\mathbb{Z} . Also, $P_\theta^{\text{in}}(m) = P_{-\theta}^{\text{out}}(m)$. For more about the incoming and outgoing partitions, see [11, §4] and [14, §7].

4.7 Statement of the index inequality

Fix a symplectic cobordism X with an admissible almost complex structure J (not necessarily generic). Continue with the notation from §4.5.

Theorem 4.15. *Suppose $C \in \mathcal{M}(\alpha^+, \alpha^-)$ is simple. Then*

$$\text{ind}(C) \leq I(C) - 2\delta(C). \quad (4.4)$$

Equality holds only if $\{q_{i,k}^+\} = P_{\alpha_i^+}^{\text{out}}(m_i^+)$ for each i , and $\{q_{j,k}^-\} = P_{\alpha_j^-}^{\text{in}}(m_j^-)$ for each j .

The proof of Theorem 4.15 has three ingredients. The first ingredient is the relative adjunction formula (4.1), which implies that the index inequality (4.4) is equivalent to the writhe bound

$$w_\tau(C) \leq \mu_\tau(C) - \mu_\tau^0(C). \quad (4.5)$$

The second ingredient is an analytic bound on the writhe $w_\tau(C)$, and the third ingredient is a combinatorial inequality. We now explain these.

4.8 The analytic writhe bound

Fix an embedded Reeb orbit γ in Y_+ at which C has positive ends of multiplicities q_1, \dots, q_n with total multiplicity m . These ends determine a braid ζ around γ with components ζ_1, \dots, ζ_n , where ζ_i has q_i strands. The braid ζ is the intersection of C with $\{R\} \times N$, where $R \gg 0$ and $N \subset Y_+$ is a small tubular neighborhood of γ .

Now fix a trivialization τ of ξ over γ . We have the following two key analytic lemmas about the writhes and linking numbers of the braids ζ_i . To simplify notation, write

$$\rho_i := \left\lfloor \frac{\text{CZ}_\tau(\gamma^{q_i})}{2} \right\rfloor.$$

Lemma 4.16. *Let $i \in \{1, \dots, n\}$. Then*

$$w_\tau(\zeta_i) \leq \rho_i(q_i - 1). \quad (4.6)$$

Equality holds only if:

- (i) *If γ is positive hyperbolic, then $q_i = 1$.*
- (ii) *If γ is negative hyperbolic, then q_i is odd or $q_i = 2$.*

(The inequality (4.6) can sometimes be improved when ρ_i and q_i have a common factor. The necessary conditions (i) and (ii) for equality are a special case of this improvement.)

Lemma 4.17. *Let $i, j \in \{1, \dots, n\}$ be distinct. Then*

$$\ell_\tau(\zeta_i, \zeta_j) \leq \max(q_i \rho_j, q_j \rho_i).$$

These lemmas were proved in [11, §6] in an easier setting^{8 9}. The asymptotic analysis necessary to carry them over to the present setting was done by Siefring [21, Theorems 2.2 and 2.3]. Combining the above two lemmas and using equation (2.9), we obtain the following bound on the writhe of ζ :

Lemma 4.18.

$$w_\tau(\zeta) \leq \sum_{i,j=1}^n \max(q_i \rho_j, q_j \rho_i) - \sum_{i=1}^n \rho_i. \quad (4.7)$$

Equality holds only if conditions (i) and (ii) in Lemma 4.16 hold.

To understand the right hand side of (4.7), we will shortly prove the following combinatorial lemma:

Lemma 4.19.

$$\sum_{i,j=1}^n \max(q_i \rho_j, q_j \rho_i) - \sum_{i=1}^n \rho_i \leq \sum_{k=1}^m \text{CZ}_\tau(\gamma^k) - \sum_{i=1}^n \text{CZ}_\tau(\gamma^{q_i}). \quad (4.8)$$

Equality holds if and only if:

(iii) *If γ is negative hyperbolic, then all q_i 's are even, except that one q_i might equal 1.*

(iv) *If γ is elliptic with monodromy angle θ , then $(q_1, \dots, q_n) = P_\theta^{\text{out}}(m)$.*

Granted this lemma, combining it with Lemma 4.18 gives

Lemma 4.20.

$$w_\tau(\zeta) \leq \sum_{k=1}^m \text{CZ}_\tau(\gamma^k) - \sum_{i=1}^n \text{CZ}_\tau(\gamma^{q_i}),$$

with equality only if $(q_1, \dots, q_k) = P_\gamma^{\text{out}}(m)$.

This lemma implies Theorem 4.15, because combining these inequalities for all the orbits in α^+ , along with analogous inequalities for the orbits in α^- , shows that the inequality (4.5) holds, with equality only under the conditions stipulated in Theorem 4.15.

⁸Note that [11, §6] discusses negative ends instead of positive ends, but this is completely analogous.

⁹The proof of Lemma 4.17 in [11] assumed that neither of the positive ends of C corresponding to ζ_i or ζ_j is “trivial”, i.e. of the form $[0, \infty) \times \gamma$. But this assumption is easily dropped: if the i end is trivial, then $q_i = 1$ and $\ell_\tau(\zeta_i, \zeta_j) = \eta_\tau(\zeta_j)$, and a fundamental winding number bound from [10] asserts that $\eta_\tau(\zeta_j) \leq \rho_j$.

4.9 Proof of the combinatorial lemma 4.19

The proof of Lemma 4.19 is easy when γ is positive hyperbolic, because by (2.2) we can choose the trivialization τ so that $\text{CZ}_\tau(\gamma^k) = 0$ for all k , and then both sides of the inequality (4.8) vanish.

If γ is negative hyperbolic, then we can choose the trivialization τ so that $\text{CZ}_\tau(\gamma^k) = k$. It is convenient to order the q_i 's so that q_1, \dots, q_k are odd, q_{k+1}, \dots, q_n are even, and $q_1 \geq \dots \geq q_k$. Then a straightforward computation shows that the inequality (4.8) is equivalent to

$$\sum_{i=1}^k \left(1 - i + \frac{1 - q_i}{2} \right) \leq 0.$$

Lemma 4.19 follows immediately in this case.

Finally, if γ is elliptic with monodromy angle θ , then by equation (2.3), Lemma 4.19 reduces to the following lemma. This was proved in [11], but we will give a new and more transparent proof here.

Lemma 4.21. *Let θ be an irrational number, and let q_1, \dots, q_n be positive integers with $m := \sum_{i=1}^n q_i$. Then*

$$\sum_{i,j=1}^n \max(q_i \lfloor q_j \theta \rfloor, q_j \lfloor q_i \theta \rfloor) \leq 2 \sum_{k=1}^m \lfloor k \theta \rfloor - \sum_{i=1}^n \lfloor q_i \theta \rfloor + m - n.$$

Equality holds if and only if $(q_1, \dots, q_n) = P_\theta^{\text{out}}(m)$.

Proof. Order the q_i 's so that

$$\frac{\lfloor q_1 \theta \rfloor}{q_1} \geq \frac{\lfloor q_2 \theta \rfloor}{q_2} \geq \dots \geq \frac{\lfloor q_n \theta \rfloor}{q_n}. \quad (4.9)$$

Let Λ denote the rightward-pointing polygonal path in the plane connecting the lattice points

$$\sum_{i=1}^j (q_i, \lfloor q_i \theta \rfloor), \quad j = 0, 1, \dots, n$$

by line segments. Let R denote the region in the plane enclosed by the path Λ , the horizontal line from the origin to $(\sum_{i=1}^n q_i, 0)$, and the vertical line from $(\sum_{i=1}^n q_i, 0)$ to $\sum_{i=1}^n (q_i, \lfloor q_i \theta \rfloor)$. Let A denote the area of the region R , let L denote the number of lattice points in R (including the boundary), and let B denote the number of lattice points on the boundary of R .

By our ordering convention (4.9), the left side of the inequality we want to prove is given by

$$\begin{aligned} \sum_{i,j=1}^n \max(q_i \lfloor q_j \theta \rfloor, q_j \lfloor q_i \theta \rfloor) &= \sum_{i=1}^n \lfloor q_i \theta \rfloor \left(q_i + 2 \sum_{j=i+1}^n q_j \right) \\ &= 2A, \end{aligned} \quad (4.10)$$

where the area of R is computed by cutting it into rectangles and triangles of height $\lfloor q_i \theta \rfloor$ and base q_j . On the other hand, Pick's formula for the area of a lattice polygon tells us that

$$2A = 2L - B - 2. \quad (4.11)$$

By dividing up the lattice points in R into vertical lines, we find that

$$L \leq 1 + \sum_{k=1}^m (\lfloor k\theta \rfloor + 1), \quad (4.12)$$

with equality if and only if the polygonal paths Λ and $\Lambda_\theta^{\text{out}}(m)$ have the same image. Finally, the number of boundary lattice points satisfies

$$B \geq m + n + \sum_{i=1}^n \lfloor q_i \theta \rfloor, \quad (4.13)$$

with equality if and only if none of the edge vectors $(q_i, \lfloor q_i \theta \rfloor)$ of the path Λ is divisible in \mathbb{Z}^2 . The lemma follows immediately by combining (4.10)–(4.13). \square

5 ECH index of unions and multiple covers

5.1 Statement of the result

As in §4, fix a symplectic cobordism X with an admissible almost complex structure J . The main result of this section is the following inequality regarding the ECH index of the union of two holomorphic curves.

Theorem 5.1. *If C and C' are holomorphic curves in X , then*

$$I(C \cup C') \geq I(C) + I(C') + 2C \cdot C'.$$

Here $C \cdot C'$ is an “intersection number” of C and C' defined below.

Note that C and C' are not assumed to be simple, irreducible, or distinct.

Definition 5.2. If C and C' are simple curves in X with no irreducible component in common, define $C \cdot C' \in \mathbb{Z}$ to be the algebraic count of intersections of C and C' . This is well-defined, because it follows from [21, Cor. 2.5] that there are only finitely many intersections. Also each intersection counts positively [17].

Now for the slightly nonstandard part of the definition.

Definition 5.3. If C is a simple, irreducible holomorphic curve in X , define

$$C \cdot C := \frac{1}{2}(2g(C) - 2 + \text{ind}(C) + h(C) + 4\delta(C)) \in \frac{1}{2}\mathbb{Z}.$$

Here $g(C)$ denotes the genus of C , and $h(C)$ denotes the number of ends of C at hyperbolic Reeb orbits.

Example 5.4. If C is closed, then it follows from (4.2) and (4.3) that $C \cdot C$ equals the usual homological intersection number $[C] \cdot [C]$.

Remark 5.5. If J is generic so that $\text{ind}(C) \geq 0$, then clearly $C \cdot C \geq -1$. In a symplectization the situation is better (without any genericity assumption on J):

Proposition 5.6. *If X is a symplectization, if C is a simple, irreducible holomorphic curve in X , and if C is not a cylinder $\mathbb{R} \times \gamma$ where γ is an elliptic Reeb orbit, then $C \cdot C \geq 0$.*

Proof. If C is a cylinder $\mathbb{R} \times \gamma$ where γ is a hyperbolic Reeb orbit then it follows immediately from the definition that $C \cdot C = 0$. If C is not a cylinder $\mathbb{R} \times \gamma$, then a stronger inequality than $C \cdot C \geq 0$ is known. Namely, if $h_+(C)$ denotes the number of ends of C at *positive* hyperbolic orbits only, then

$$2g(C) - 2 + \text{ind}(C) + h_+(C) \geq 0,$$

cf. [10] and [26, Prop. 4.1]. The proof of this inequality uses a linearized version of positivity of intersections of C with its translates in the \mathbb{R} direction. \square

Definition 5.7. If C is a union of d_a -fold covers of distinct simple irreducible curves C_a , and if C' is a union of d'_b -fold covers of distinct simple irreducible curves C'_b , then

$$C \cdot C' := \sum_a \sum_b d_a d'_b C_a \cdot C'_b.$$

This completes the statement of Theorem 5.1.

Example 5.8. The inequality (1.2) is a special case of Theorem 5.1 in which X is a symplectization and C' is a union of \mathbb{R} -invariant cylinders and the image of C contains no \mathbb{R} -invariant cylinder. This was proved in [11, Prop. 7.1]. In this case one also obtains necessary conditions for equality in terms of the incoming and outgoing partitions; for more about these conditions see [14, Lem. 7.28].

Remark 5.9. The proof of Theorem 5.1 will show that in some cases one can obtain a stronger inequality, in which $C \cdot C'$ is replaced by a more elaborate notion of intersection number involving additional contributions from the ends. Some related notions of intersection number are discussed in [27, §4], using work of Siefring [22].

5.2 Proof of Theorem 5.1

The proof of Theorem 5.1 will use the following combinatorial lemma.

Lemma 5.10. *Let γ be a Reeb orbit, let q_1, \dots, q_n be positive integers with $m := \sum_{i=1}^n q_i$, let $q'_1, \dots, q'_{n'}$ be positive integers with $m' := \sum_{j=1}^{n'} q'_j$, and let $\rho_i := \lfloor \text{CZ}_\tau(\gamma^{q_i})/2 \rfloor$ and $\rho'_j := \lfloor \text{CZ}_\tau(\gamma^{q'_j})/2 \rfloor$. Then*

$$2 \sum_{i=1}^n \sum_{j=1}^{n'} \max(q_i \rho'_j, q'_j \rho_i) \leq \left(\sum_{k=1}^{m+m'} - \sum_{k=1}^m - \sum_{k=1}^{m'} \right) \text{CZ}_\tau(\gamma^k). \quad (5.1)$$

Proof. We consider two cases.

Case 1: If γ is hyperbolic, then by equation (2.2), there is an integer l such that $\text{CZ}_\tau(\gamma^k) = kl$ for all k . Then the inequality (5.1) is equivalent to

$$2 \sum_{i=1}^n \sum_{j=1}^{n'} \max \left(q_i \left\lfloor \frac{lq'_j}{2} \right\rfloor, q'_j \left\lfloor \frac{lq_i}{2} \right\rfloor \right) \leq lmm'.$$

But this inequality is obvious because

$$\max \left(q_i \left\lfloor \frac{lq'_j}{2} \right\rfloor, q'_j \left\lfloor \frac{lq_i}{2} \right\rfloor \right) \leq \frac{lq_i q'_j}{2}. \quad (5.2)$$

Case 2: If γ is elliptic with monodromy angle θ , then by equation (2.3), the inequality (5.1) is equivalent to

$$\sum_{i=1}^n \sum_{j=1}^{n'} \max(q_i \lfloor q'_j \theta \rfloor, q'_j \lfloor q_i \theta \rfloor) \leq \left(\sum_{k=1}^{m+m'} - \sum_{k=1}^m - \sum_{k=1}^{m'} \right) \lfloor k\theta \rfloor. \quad (5.3)$$

We prove this by induction on $n + n'$. Without loss of generality, $n \geq 1$, and $q_n^{-1} \lfloor q_n \theta \rfloor$ is the smallest number in the set $\{q_i^{-1} \lfloor q_i \theta \rfloor\} \cup \{q_j'^{-1} \lfloor q_j' \theta \rfloor\}$. Then

$$\begin{aligned} \sum_{j=1}^{n'} \max(q_n \lfloor q_j' \theta \rfloor, q_j' \lfloor q_n \theta \rfloor) &= q_n \sum_{j=1}^{n'} \lfloor q_j' \theta \rfloor \\ &\leq q_n \lfloor m' \theta \rfloor \\ &\leq \sum_{k=m-q_n+1}^m (\lfloor (k+m') \theta \rfloor - \lfloor k \theta \rfloor). \end{aligned} \tag{5.4}$$

This reduces (5.3) to the corresponding inequality with n decreased by 1 and q_n removed, so we are done by induction. \square

Proof of Theorem 5.1. Let C_1, \dots, C_r denote the distinct irreducible simple curves that appear in the image of either C or C' . Thus C consists of a d_a -fold cover of C_a for each $a = 1, \dots, r$, and C' consists of a d'_a -fold cover of C_a for each a , for some $d_a, d'_a \geq 0$.

To prove the theorem, we begin by reducing to a local statement around each Reeb orbit. Since c_τ is additive, and Q_τ is quadratic as in (3.11), we have

$$I(C) = \sum_{a=1}^r d_a c_\tau(C_a) + \sum_{a,b=1}^r d_a d_b Q_\tau(C_a, C_b) + \mu_\tau(C).$$

Adding the analogous equation for $I(C')$, and subtracting the result from the analogous equation for $I(C \cup C')$, we obtain

$$I(C \cup C') - I(C) - I(C') = 2 \sum_{a,b=1}^r d_a d'_b Q_\tau(C_a, C_b) + \mu_\tau(C \cup C') - \mu_\tau(C) - \mu_\tau(C').$$

By the definition of Q , if $a \neq b$ then

$$Q_\tau(C_a, C_b) = C_a \cdot C_b - \ell_\tau(C_a, C_b).$$

On the other hand, by the relative adjunction formula (4.1) and the index formula (4.3), we have

$$Q_\tau(C_a) = C_a \cdot C_a + \frac{1}{2} (e(C_a) - \mu_\tau^0(C_a) - 2w_\tau(C_a)),$$

where $e(C_a)$ denotes the number of ends of C_a at elliptic Reeb orbits. Putting

this all together gives

$$\begin{aligned}
I(C \cup C') - I(C) - I(C') - 2C \cdot C' &= \mu_\tau(C \cup C') - \mu_\tau(C) - \mu_\tau(C') \\
&\quad - 2 \sum_{a \neq b} d_a d'_b \ell_\tau(C_a, C_b) \\
&\quad + \sum_{a=1}^r d_a d'_a (e(C_a) - \mu_\tau^0(C_a) - 2w_\tau(C_a)).
\end{aligned} \tag{5.5}$$

We need to prove that the right side of this equation is nonnegative.

Let γ be a Reeb orbit in Y_+ at which some of the curves C_a have positive ends. For $a = 1, \dots, r$, let $n_a \geq 0$ denote the number of positive ends of C_a at γ , let $q_{a,1}, \dots, q_{a,n_a}$ denote the multiplicities of these ends, let $m_a := \sum_{i=1}^{n_a} q_{a,i}$, and let ζ_a denote the braid around γ corresponding to these ends. Also let $M := \sum_{a=1}^r d_a m_a$ and $M' := \sum_{a=1}^r d'_a m_a$. Define ε to equal 1 if γ is elliptic and 0 if γ is hyperbolic. Then it is enough to show that

$$\begin{aligned}
\left(\sum_{k=1}^{M+M'} - \sum_{k=1}^M - \sum_{k=1}^{M'} \right) \text{CZ}_\tau(\gamma^k) &\geq 2 \sum_{a \neq b} d_a d'_b \ell_\tau(\zeta_a, \zeta_b) \\
&\quad + \sum_{a=1}^r d_a d'_a \left(-\varepsilon n_a + \sum_{i=1}^{n_a} \text{CZ}_\tau(\gamma^{q_{a,i}}) + 2w_\tau(\zeta_a) \right).
\end{aligned} \tag{5.6}$$

(We also need to prove an analogous inequality for the negative ends, but this is completely symmetric.) To prove the inequality (5.6), we consider two cases.

Case 1: Suppose γ is elliptic with monodromy angle θ . As usual, write $\rho_{a,i} := \lfloor \text{CZ}_\tau(\gamma^{q_{a,i}})/2 \rfloor$. For $a, b = 1, \dots, r$, introduce the notation

$$f(a, b) := \sum_{i=1}^{n_a} \sum_{j=1}^{n_b} \max(q_{a,i} \rho_{b,j}, q_{b,j} \rho_{a,i})$$

By Lemma 4.17, for $a \neq b$ we have

$$\ell_\tau(\zeta_a, \zeta_b) \leq f(a, b).$$

Since $\text{CZ}_\tau(\gamma^k)$ is odd, Lemma 4.18 implies that

$$2w_\tau(\zeta_a) \leq 2f(a, a) - \sum_{i=1}^{n_a} \text{CZ}_\tau(\gamma^{q_{a,i}}) + n_a.$$

So to prove (5.6) in this case, it is enough to show that

$$2 \sum_{a,b=1}^r d_a d'_b f(a,b) \leq \left(\sum_{k=1}^{M+M'} - \sum_{k=1}^M - \sum_{k=1}^{M'} \right) \text{CZ}_\tau(\gamma^k).$$

But this follows by applying Lemma 5.10 to the list consisting of the numbers $q_{a,i}$ repeated d_a times, and the list consisting of the numbers $q_{a,i}$ repeated d'_a times.

Case 2: Suppose γ is hyperbolic. For $a \neq b$, by Lemmas 4.17 and 5.10 we have

$$2\ell_\tau(\zeta_a, \zeta_b) \leq \left(\sum_{k=1}^{m_a+m_b} - \sum_{k=1}^{m_a} - \sum_{k=1}^{m_b} \right) \text{CZ}_\tau(\gamma^k).$$

And by Lemma 4.20, we have

$$w_\tau(\zeta_a) \leq \sum_{k=1}^{m_a} \text{CZ}_\tau(\gamma^k) - \sum_{i=1}^{n_a} \text{CZ}_\tau(\gamma^{q_{a,i}}).$$

So to prove the inequality (5.6) in this case, it is enough to show that

$$\begin{aligned} \left(\sum_{k=1}^{M+M'} - \sum_{k=1}^M - \sum_{k=1}^{M'} \right) \text{CZ}_\tau(\gamma^k) &\geq \sum_{a \neq b} d_a d'_b \left(\sum_{k=1}^{m_a+m_b} - \sum_{k=1}^{m_a} - \sum_{k=1}^{m_b} \right) \text{CZ}_\tau(\gamma^k) \\ &\quad + \sum_{a=1}^r d_a d'_a \left(- \sum_{i=1}^{n_a} \text{CZ}_\tau(\gamma^{q_{a,i}}) + 2 \sum_{k=1}^{m_a} \text{CZ}_\tau(\gamma^k) \right). \end{aligned}$$

But a straightforward computation using equation (2.2) shows that this last inequality is always an equality. \square

6 The ECH index and the Euler characteristic

To put the previous results in perspective, we now introduce a natural variant of the ECH index, which bounds the negative Euler characteristic of holomorphic curves, and which gives rise to a relative filtration on ECH, or any other kind of contact homology of a contact 3-manifold.

6.1 Definition of J_0 , J_+ , and J_-

Notation 6.1. In a 3-manifold with a stable Hamiltonian structure (in which all Reeb orbits are assumed nondegenerate as usual), if $\alpha = \{(\alpha_i, m_i)\}$ is an orbit set and if τ is a trivialization of ξ over the α_i 's, define

$$\mu'_\tau(\alpha) := \sum_i \sum_{k=1}^{m_i-1} \text{CZ}_\tau(\alpha_i^k).$$

This differs from the quantity $\mu_\tau(\alpha)$ defined in §2.8 only in that m_i there is replaced by $m_i - 1$ here.

Now let X be a symplectic cobordism from Y_+ to Y_- , and continue with the notation from §4.2.

Definition 6.2. Let α^+ be an orbit set in Y_+ and let α^- be an orbit set in Y_- such that $[\alpha^+] = [\alpha^-] \in H_1(X)$, and let $Z \in H_2(X, \alpha^+, \alpha^-)$. Define

$$J_0(\alpha^+, \alpha^-, Z) := -c_\tau(Z) + Q_\tau(Z) + \mu'_\tau(\alpha^+) - \mu'_\tau(\alpha^-).$$

Here τ is a trivialization of ξ_+ over the orbits in α^+ and of ξ_- over the orbits in α^- . The definition of J_0 differs from that of the ECH index I only in that the sign of c_τ is switched, and μ_τ is replaced by μ'_τ . The usual considerations using equations (2.6), (2.11), and (2.4) show that J_0 does not depend on τ .

Example 6.3. Suppose that C is an embedded holomorphic curve in X , whose ends are at distinct Reeb orbits with multiplicity 1. Then $w_\tau(C) = 0$ and $\mu'_\tau(C) = 0$, so it follows from the relative adjunction formula (4.1) that

$$J_0(C) = -\chi(C).$$

Two variants of J_0 are also of interest.

Definition 6.4. If $\alpha = \{(\alpha_i, m_i)\}$ is an orbit set, define

$$|\alpha| := \sum_i \begin{cases} 1, & \alpha_i \text{ elliptic,} \\ m_i, & \alpha_i \text{ positive hyperbolic,} \\ \lceil m_i/2 \rceil, & \alpha_i \text{ negative hyperbolic.} \end{cases}$$

Now define

$$\begin{aligned} J_+(\alpha^+, \alpha^-, Z) &:= J_0(\alpha^+, \alpha^-, Z) + |\alpha^+| - |\alpha^-|, \\ J_-(\alpha^+, \alpha^-, Z) &:= J_0(\alpha^+, \alpha^-, Z) - |\alpha^+| + |\alpha^-|. \end{aligned}$$

6.2 Properties of J_0 , J_+ , and J_- in symplectizations

Suppose now that X is the symplectization of a closed oriented 3-manifold Y with a stable Hamiltonian structure. Then J_0 , J_+ , and J_- have the following basic properties, which are similar to those of the ECH index I :

Proposition 6.5. *Suppose X is the symplectization of Y . Fix J to denote one of J_0 , J_+ , or J_- . Then:*

(a) (Additivity) *If $Z \in H_2(Y, \alpha, \beta)$ and $W \in H_2(Y, \beta, \gamma)$ then*

$$J(\alpha, \beta, Z) + J(\beta, \gamma, W) = J(\alpha, \gamma, W).$$

(b) (Ambiguity) If $Z, Z' \in H_2(Y, \alpha, \beta)$ where $[\alpha] = [\beta] = \Gamma \in H_1(Y)$ then

$$J(\alpha, \beta, Z) - J(\alpha, \beta, Z') = \langle -c_1(\xi) + 2\text{PD}(\Gamma), Z - Z' \rangle.$$

(c) (Absolute version) For each orbit set $\alpha = \{(\alpha_i, m_i)\}$, there is a homotopy class of oriented 2-plane fields $J(\alpha) \in \mathcal{P}(Y)$ such that:

(i) $J(\alpha)$ is obtained by modifying ξ by a canonical manner (up to homotopy, depending only on m_i) in disjoint tubular neighborhoods of each α_i .

(ii) $\mathfrak{s}(J(\alpha)) = \mathfrak{s}(\xi) - \text{PD}([\alpha])$.

(iii) If α and β are orbit sets with $[\alpha] = [\beta] = \Gamma$ then $J(\alpha, \beta, Z) \equiv J(\alpha) - J(\beta)$ in $\mathbb{Z}/d(-c_1(\xi) + 2\text{PD}(\Gamma))$.

Proof. One mimics the proofs of the corresponding properties of I . For part (c), one replaces equation (3.9) by

$$J_0(\alpha) := P'(L) - \sum_i w_{\tau_i}(\zeta_i) + \mu'_\tau(\alpha),$$

where $P'(L)$ was defined in Remark 3.5. □

The culminating result to be proved in this section is:

Theorem 6.6. *Suppose X is the symplectization of a contact 3-manifold Y , with an admissible almost complex structure. Then every holomorphic curve C in X satisfies $J_+(C) \geq 0$.*

Note that the almost complex structure is not assumed to be generic, and the holomorphic curve C is not assumed to be simple or irreducible.

Remark 6.7. This theorem implies that the differential ∂ in the embedded contact homology (or any other kind of contact homology) of a contact 3-manifold can be decomposed as $\partial = \partial_0 + \partial_1 + \dots$ where ∂_k denotes the contribution from holomorphic curves C with $J_+(C) = k$. By the additivity of J_+ , the identity $\partial^2 = 0$ can be refined to $\partial_0^2 = 0$, $\partial_0\partial_1 + \partial_1\partial_0$, etc. However it is not clear if this leads to new topological invariants, except perhaps in some special situations, because in general maps induced by cobordisms might include contributions from curves with J_+ negative.

Example 6.8. A tool that was used in [13] to help compute the embedded contact homology of T^3 turns out to be a special case of the relative filtration J_+ . Namely, for the contact forms on T^3 considered in [13], $J_+(\alpha, \beta, Z)$ equals $I(\alpha, \beta, Z)$ plus the number of hyperbolic orbits in α minus the number of hyperbolic orbits in β . All curves that contribute to the ECH differential in this case have $J_+ = 2$. Thus $2I - J_+$ defines a second grading which is preserved by the differential, and this is what appears in [13, Def. 5.1].

6.3 Lower bounds on J_0 in the general case

Theorem 4.15 has the following analogue for J_0 .

Proposition 6.9. *Let X be a symplectic cobordism from Y_+ and Y_- , let α^+ be an orbit set in Y_+ , and let α^- be an orbit set in Y_- . Suppose $C \in \mathcal{M}(\alpha^+, \alpha^-)$ is simple and irreducible and has genus g . Then*

$$J_0(C) \geq 2(g - 1 + \delta(C)) + \sum_{\gamma} \begin{cases} 2n_{\gamma} - 1, & \gamma \text{ elliptic,} \\ m_{\gamma}, & \gamma \text{ positive hyperbolic,} \\ \frac{m_{\gamma} + n_{\gamma}^{\text{odd}}}{2}, & \gamma \text{ negative hyperbolic.} \end{cases} \quad (6.1)$$

Here the sum is over all embedded Reeb orbits γ in Y_+ or Y_- at which C has ends; m_{γ} denotes the total multiplicity of the ends of C at γ ; n_{γ} denotes the number of ends of C at γ ; and n_{γ}^{odd} denotes the number of ends of C at γ with odd multiplicity¹⁰.

Proof. Let n denote the number of ends of C . The relative adjunction formula (4.1) implies that

$$J_0(C) = 2g - 2 + n + \mu'_{\tau}(C) - w_{\tau}(C) + 2\delta(C).$$

Thus it is enough to show that $n + \mu'_{\tau}(C) - w_{\tau}(C)$ is greater than or equal to the sum over γ in (6.1).

To prove this last inequality, we can assume without loss of generality (as will be clear from the argument below) that there is a single embedded Reeb orbit γ in Y_+ such that all ends of C are positive ends at γ . Thus $\alpha^- = \emptyset$, and we can write $\alpha^+ = \{(\gamma, m)\}$. Let q_1, \dots, q_n denote the multiplicities of the positive ends of C at γ , so in particular $\sum_{i=1}^n q_i = m$. Let ζ_1, \dots, ζ_n denote the corresponding braids around γ , and let $\zeta := \bigcup_i \zeta_i$. Then we need to show that

$$n + \sum_{k=1}^{m-1} CZ_{\tau}(\gamma^k) - w_{\tau}(\zeta) \geq \begin{cases} 2n - 1, & \gamma \text{ elliptic,} \\ m, & \gamma \text{ positive hyperbolic,} \\ \frac{m + n_{\text{odd}}}{2}, & \gamma \text{ negative hyperbolic.} \end{cases} \quad (6.2)$$

Here n_{odd} denotes the number of odd q_i 's.

Case 1: γ is elliptic. In this case the proof of (6.2) follows the proof of the inequality in Lemma 4.20. However instead of the combinatorial inequality in Lemma 4.21, one needs the slightly stronger inequality

$$\sum_{i,j=1}^n \max(q_i \lfloor q_j \theta \rfloor, q_j \lfloor q_i \theta \rfloor) \leq 2 \sum_{k=1}^{m-1} \lfloor k\theta \rfloor + \sum_{i=1}^n \lfloor q_i \theta \rfloor + m - n.$$

¹⁰When X is a symplectization, Y_+ and Y_- are still regarded as distinct, i.e. the positive and negative ends are to be counted in separate summands in (6.1).

This is proved the same way as Lemma 4.21, but with the inequality (4.12) replaced by the equally obvious inequality

$$L \leq 2 + \sum_{k=1}^{m-1} ([k\theta] + 1) + \sum_{i=1}^n [q_i\theta].$$

Case 2: γ is positive hyperbolic. In this case Lemma 4.16 can be improved to

$$w_\tau(\zeta_i) \leq (\rho_i - 1)(q_i - 1).$$

This can be proved by the arguments in [11, Lem. 6.8], using [21, Thm. 2.3] to provide the necessary asymptotic analysis in the present setting. Hence the writhe bound in Lemma 4.20 can be improved to

$$w_\tau(\zeta) \leq \sum_{k=1}^m \text{CZ}_\tau(\gamma^k) - \sum_{i=1}^n \text{CZ}_\tau(\gamma^{q_i}) - \sum_{i=1}^n (q_i - 1).$$

So to prove (6.2) in this case it is enough to show that

$$n - \text{CZ}_\tau(\gamma^m) + \sum_{i=1}^n \text{CZ}_\tau(\gamma^{q_i}) + \sum_{i=1}^n (q_i - 1) \geq m.$$

But this is an equality, because by (2.2), all the CZ terms cancel.

Case 3: γ is negative hyperbolic. Choose the trivialization τ so that $\text{CZ}(\tau^k) = k$. In this case Lemma 4.16 can be improved to

$$w_\tau(\zeta_i) \leq \left\lceil \frac{(q_i - 1)^2}{2} \right\rceil,$$

again by the arguments in [11, Lem. 6.8] with the help of [21, Thm. 2.3]. So together with Lemma 4.17, we obtain

$$w_\tau(\zeta) \leq \sum_{i=1}^n \left\lceil \frac{(q_i - 1)^2}{2} \right\rceil + 2 \sum_{i=1}^n \sum_{j=i+1}^n \max\left(q_i \left\lfloor \frac{q_j}{2} \right\rfloor, q_j \left\lfloor \frac{q_i}{2} \right\rfloor\right).$$

To simplify this, order the q_i 's so that $q_1, \dots, q_{n_{\text{odd}}}$ are odd and $q_1 \geq \dots \geq q_{n_{\text{odd}}}$. A straightforward calculation then deduces from the above inequality that

$$n + \sum_{k=1}^{m-1} \text{CZ}_\tau(\gamma^k) - w_\tau(\zeta) \geq \frac{m + n_{\text{odd}}}{2} + \sum_{j=1}^{n_{\text{odd}}} (j - 1)q_j.$$

The inequality (6.2) follows. \square

Corollary 6.10. *If C is a simple holomorphic curve as above, then*

$$-\chi(C) \leq J_0(C) - 2\delta(C).$$

Proof. We just have to check that for each γ at which C has ends, the corresponding summand in (6.1) is at least n_γ . But this is easy. (For the negative hyperbolic case, note that since each end has multiplicity at least one if odd and at least two if even, we have $m_\gamma \geq n_\gamma^{odd} + 2(n_\gamma - n_\gamma^{odd})$.) The argument works just as well if C is not irreducible, as long as it is simple. \square

By a similar but even easier argument, we have:

Corollary 6.11. *If $C \in \mathcal{M}(\alpha^+, \alpha^-)$ is a simple irreducible holomorphic curve as above, then*

$$J_0(C) \geq 2(g - 1 + \delta(C)) + |\alpha^+| + |\alpha^-|,$$

or equivalently

$$J_\pm(C) \geq 2(g - 1 + |\alpha^\pm| + \delta(C)).$$

Remark 6.12. This last inequality shows that J_+ is similar to the relative filtration on the symplectic field theory [6] of a contact manifold given by genus plus number of positive ends minus one.

Remark 6.13. The inequality (6.1) implies the index inequality (4.4). One can see this by adding the index formula (4.3) and then arguing as in the proofs of the above corollaries. In particular, if $\text{ind}(C) = I(C)$ (e.g. if C is a curve in the symplectization of a contact manifold that contributes to the ECH differential and does not contain trivial cylinders), then the inequality (6.1) is sharp.

6.4 J_0 of unions and multiple covers

As usual, let X be a symplectic cobordism from Y_+ to Y_- with an admissible almost complex structure.

Proposition 6.14. *If C and C' are holomorphic curves in X , then*

$$J_0(C \cup C') \geq J_0(C) + J_0(C') + 2C \cdot C' + E + N,$$

where¹¹:

- E denotes the number of elliptic Reeb orbits in Y_+ or Y_- at which both C and C' have ends.
- N denotes the number of negative hyperbolic orbits γ in Y_+ or Y_- such that the total multiplicity of the ends of C at γ and the total multiplicity of the ends of C' at γ are both odd.

¹¹When X is a symplectization, Y_+ and Y_- are still regarded as distinct in the definition of E and N .

Proof. One copies the proof of Theorem 5.1 with minor modifications. In particular, the same calculation as before shows that equation (5.5) holds with I replaced by J_0 and with μ_τ replaced by μ'_τ . To prove that the right hand side of this modified equation (5.5) is at least $E + N$, one follows the proof of (5.6), but replacing Lemma 5.10 with Lemma 6.15 below. \square

Lemma 6.15. *Under the assumptions of Lemma 5.10, we have*

$$2 \sum_{i=1}^n \sum_{j=1}^{n'} \max(q_i \rho'_j, \rho_i q'_j) \leq \left(\sum_{k=1}^{m+m'-1} - \sum_{k=1}^{m-1} - \sum_{k=1}^{m'-1} \right) \text{CZ}_\tau(\gamma^k). \quad (6.3)$$

If γ is elliptic and $m, m' > 0$, or if γ is negative hyperbolic and both m and m' are odd, then the inequality is strict.

Proof. We slightly modify the proof of Lemma 5.10 as follows. If γ is hyperbolic, then the inequality (6.3) is equivalent to (5.1) because the right hand sides are equal by (2.2). If γ is negative hyperbolic and both m and m' are odd, then there is a pair (i, j) such that q_i and q'_j are both odd so that the inequality (5.2) is strict, so (5.1) is strict.

Now suppose γ is elliptic with monodromy angle θ . Without loss of generality $m, m' > 0$. By equation (2.3), the right hand side of (6.3) minus the right hand side of (5.1) equals

$$1 - 2 \left(\lfloor (m + m')\theta \rfloor - \lfloor m\theta \rfloor - \lfloor m'\theta \rfloor \right) \in \{-1, 1\}.$$

If this is 1 then we are done. If this is -1 , then equality does not hold in (5.4), so the two sides of (5.1) differ by at least 2 and we are also done. \square

6.5 The relative filtration J_+

Proof of Theorem 6.6. Since $J_+(C)$ depends only on the relative homology class of C , we may assume that C is a union of k (not necessarily distinct) simple, irreducible holomorphic curves. We now prove the theorem by induction on k .

If C is simple and irreducible, then $J_+(C) \geq 0$ by Corollary 6.11, since the assumption that Y is a contact manifold guarantees that C has at least one positive end. Also, if C is any multiple cover of a cylinder $\mathbb{R} \times \gamma$ where γ is a Reeb orbit, then $J_+(C) = 0$ by definition.

To complete the induction, it is enough to show that if $C \in \mathcal{M}(\alpha, \beta)$ and $C' \in \mathcal{M}(\alpha', \beta')$ satisfy $J_+(C), J_+(C') \geq 0$, and if the images of C and C' do not have a cylinder $\mathbb{R} \times \gamma$ in common, then $J_+(C \cup C') \geq 0$. Note that $C \cdot C' \geq 0$ by intersection positivity and Proposition 5.6. So by Proposition 6.14,

$$J_+(C \cup C') \geq E + N + (|\alpha\alpha'| - |\alpha| - |\alpha'|) - (|\beta\beta'| - |\beta| - |\beta'|).$$

Now write

$$E = E_+ + E_-, \quad N = N_+ + N_-,$$

where E_+ denotes the number of elliptic orbits that appear in both α and α' ; E_- denotes the number of elliptic orbits that appear in both β and β' ; N_+ denotes the number of negative hyperbolic orbits that appear with odd multiplicity in both α and α' ; and N_- denotes the number of negative hyperbolic orbits that appear with odd multiplicity in both β and β' . It follows from Definition 6.4 that

$$\begin{aligned} |\alpha\alpha'| &= |\alpha| + |\alpha'| - E_+ - N_+, \\ |\beta\beta'| &= |\beta| + |\beta'| - E_- - N_-. \end{aligned}$$

Putting the above together gives $J_+(C \cup C') \geq 2(E_- + N_-) \geq 0$. \square

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