On ODEs of Type $(y')^{n-1} = a_n y^n + a_{n-1} y^{n-1} + ... + a_0$

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Abstract

We give an explicit algebraic description of local solutions to $(y')^{n-1} = g(y)$, where $g(y)$ is a polynomial in y with complex coefficients of degree n, by means of the classical theory of algebraic curves.

Introduction

In this very short paper, we will mainly be concerned with the local solutions to the ordinary differential equation

$$(y')^{n-1} = g(y) \tag{1}$$

where $g(y)$ is a polynomial in y with complex coefficients of degree n, $n \geq 2$.

For any (typically nonlinear) ODE of type $f(y, y') = 0$, where f is a polynomial in two variables, one can associate it with certain algebraic curve embedded in $\mathbb{CP}^2$ and try to find or describe local holomorphic solutions to the differential equation algebraically by manipulating with certain meromorphic functions and differentials on the curve and possibly taking some limits. This is nineteen century style mathematics which should have been known long time ago. However, except for materials on the Weierstrass $\wp$ function, I don't find any explicit reference of this in the literature. (Probably because most literature I read are mordern campared to this) The aim of the present paper is thus to try to remind ourselves a bit of this beautiful and classical nineteen century style mathematics with the example of (1). The restriction to the particular form of (1) is by no means necessary theoretically, however, (1) is one of the simplest nontrivial cases in several aspects, for example, it's rather easy to describe its real solutions from the complex ones. As the simplest illustration, in the end of the paper, we'll give an explicit algebraic geometric interpretation of solutions to $y' = y(y - b)$, which are routinely taught in freshmen calculus courses without mentioning algebraic geometry.

More explicitly, the Weierstrass $\wp$ function satisfies an ODE of the above type for $n = 3$. If we are interested in local holomorphic solutions to (1) for general n, we can try to describe the solutions with the aid of certain higher genus algebraic curves associated with (1). It turns out that when the complex roots of $g(y)$ are all distinct, all local holomorphic solutions indeed come from a certain meromorphic function on an embedded nonsingular projective curve in $\mathbb{CP}^2$ determined by (1), under a certain choice of local coordinates. This is a direct generalization of the Weierstrass $\wp$ function case (for background material on this case, one can consult standard textbooks as [1]), and in the sense of algebraic geometry, the proof essentially is no more difficult than that for the Weierstrass $\wp$ function case. Furthermore, straightforwardly, one can use this to give an algebraic geometric description of all local solutions to initial value problems of (1). And when $g(y)$ is a real polynomial, and the initial value is real, we can also describe all local real solutions of (1) easily with the aid of these local holomorphic solutions.

We will first find a local holomorphic solution to (1) in the next section, then we will briefly discuss local solutions of initial value problems of (1) in the section after that. The algebraic geometry and notations we use can be found in any standard textbook, for example, in [2].

1 A local solution to (1)

In this section, we assume all complex roots of $g(y)$ are distinct, and $n \geq 3$. Just for notational simplicity, by a scaling transformation if necessary, without loss of generality, we can assume $g(y)$ is monic. We write $g(y) = (y - a_1)(y - a_2)...(y - a_n)$. Let C denote the nonsingular algebraic curve $y^{n-1}z = (x - a_1z)(x - a_2z)...(x - a_nz)$ in $\mathbb{CP}^2$. And by points on C we refer to its complex points. We denote by ω and $\mathcal{O}$ for the canonical sheaf and structure sheaf of C, respectively. Then $\omega \cong \mathcal{O}(n-3)$ as $\mathcal{O}$ modules. Choose an isomorphism φ. Then we have the following:

Theorem 1.1. *Let P be any point of C which is not a pole of the global meromorphic function $\frac{x}{z}$. Pick a local holomorphic coordinate s near P such that $ds = \varphi(y^{n-3})$. Then there exists a complex number λ such that the function $\frac{x}{z}$, when locally regarded as a holomorphic function of λs, gives a local solution to* (1)*.*

Proof. The proof consists of simply counting zeros and poles.

For the global meromorphic function $\frac{x-a_kz}{z}$, $1 \leq k \leq n$. we have:
zeros of multiplicity $n-1$ at $[a_k : 0 : 1]$, as $v(x) = n-1$ in the corresponding local ring;
poles of degree $n-1$ at $[0 : 1 : 0]$, as $v(x) = 1$, $v(z) = v(x^n) = n$ in the corresponding local ring.
One can verify easily by working on each open set $D(x)$, $D(y)$ and $D(z)$ that these are all the zeros and poles for $\frac{x-a_kz}{z}$.
So, the global meromorphic differential $d(\frac{x-a_kz}{z})$=$d(\frac{x}{z})$ has zeros of multiplicity $n-2$ at $[a_k : 0 : 1]$, $1 \leq k \leq n$, and poles of degree n at $[0 : 1 : 0]$. clearly we have found all the poles of $d(\frac{x}{z})$, since they all come from poles of $\frac{x-a_kz}{z}$.

Moreover, the divisor of any global meromorphic differential has degree equal to the degree of the canonical divisor, which is $2g-2$, where $g = \frac{1}{2}(n-1)(n-2)$ is the genus of C. So this degree equals $(n-1)(n-2)-2$=$n(n-3)$, which tells us that $d(\frac{x}{z})$ has $n(n-3)$ more zeros than poles. On the other hand, we already know $d(\frac{x}{z})$ has exactly n poles and we have found $n(n-2)$ zeros of it. So all the zeros and poles of $d(\frac{x}{z})$ are listed above.

Next, we count zeros and poles for the global meromorphic function $\frac{y}{z}$. It has zeros of multiplicity 1 at each $[a_k : 0 : 1]$, and poles of degree n at $[0 : 1 : 0]$. Again one can easily verify these are all the zeros and poles for $\frac{y}{z}$ by checking on open sets.
Furthermore, we take the global section y^{n-3} of $\mathcal{O}(n-3)$. It has zeros of multiplicity $n-3$ at each $[a_k : 0 : 1]$, and no poles.

Now, consider the global meromorphic differential $\frac{y}{z}\varphi(y^{n-3})$. Its zeros and poles are:
poles of degree n at $[0 : 1 : 0]$;
zeros of multiplicity $n-2$ at each $[a_k : 0 : 1]$.

Comparing this with the zeros and poles of $d(\frac{x}{z})$, we see that these two meromorphic differentials have to be equal up to a constant scalar. So, there exists a complex number

λ such that $d(\frac{x}{z})=\lambda\frac{y}{z}\varphi(y^{n-3})$. We choose some local holomorphic coordinate s such that $ds=\varphi(y^{n-3})$, then this equality can be rewritten as $d(\frac{x}{z})=\frac{y}{z}d(\lambda s)$. Locally, we may regard $\frac{x}{z}$ as a holomorphic function of the complex variable λs. And This equality means it's derivative equals $\frac{y}{z}$. But $(\frac{y}{z})^n=g(\frac{x}{z})$ by the equation of C. Combining these, we see that the function $\frac{x}{z}$, when locally regarded as a holomorphic function of λs, gives a local solution to (1). □

Before we discuss initial value problems of (1), we'd like to slightly generalize the above theorem to the case $h(y')=g(y)$, where h is a polynomial of degree $n-1$. This is straightforward. By applying some linear transformation if necessary, we may assume without loss of generality that $h(t)=t^m f(t)$, where $m\geq 2$, and $f(t)$ a polynomial of degree $n-m-1$ with nonzero constant term. Let's denote $f(t)=b_{n-m-1}t^{n-m-1}+...+b_0$, and $h'(t)=t^{m-1}(c_{n-m-1}t^{n-m-1}+...+c_0)$, then we have the following:

Theorem 1.2. *(A generalization of theorem 1.1) If the curve C': $y^m(b_{n-m-1}y^{n-m-1}+...+b_0z^{n-m-1})=(x-a_1z)(x-a_2z)...(x-a_nz)$ in $\mathbb{CP}^2$ is nonsingular (which is true for generic case), Let P be any point of C' which is not a pole of the global meromorphic function $\frac{x}{z}$. Pick a local holomorphic coordinate s near P such that $ds=\varphi(y^{m-2}(c_{n-m-1}y^{n-m-1}+...+c_0z^{n-m-1}))$. Then there exists a complex number λ such that the function $\frac{x}{z}$, when locally regarded as a holomorphic function of λs, gives a local solution to $h(y')=g(y)$.*

Proof. We denote $f_1(t)=c_{n-m-1}t^{n-m-1}+...+c_0$. For simplicity, we assume all complex roots of $f_1(t)$ are of multiplicity 1. (The general case can be discussed directly with no more difficulty, or can be derived by taking a limit in some sense of this generic case.) Then, as before, the meromorphic function $\frac{x-a_kz}{z}$ has zeros of multiplicity m at $[a_k:0:1]$, and all poles of it are at $[0:1:0]$ of degree $n-1$. So the global meromorphic differential $d(\frac{x}{z})$ has zeros of multiplicity $m-1$ at $[a_k:0:1]$, and all poles of it are at $[0:1:0]$ of degree n.

Furthermore, for any root y_r of $f_1(t)$, $y_r\neq 0$ by the definition of $f(t)$, and y_r is a double root of $h(y)-h(y_r)$. This means that for any root α of $g(x)-h(y_r)$, the function $\frac{x-\alpha z}{z}$ has a double zero at $[\alpha:y_r:1]$. (note that $g(x)-h(y_r)$ can not have roots of multiplicity greater than 1, because the curve C' is nonsingular.) So $d(\frac{x-\alpha z}{z})=d(\frac{x}{z})$ has a simple zero at $[\alpha:y_r:1]$. Since $f_1(t)$ is of degree $n-m-1$, these contributes $n(n-m-1)$ many zeros to $d(\frac{x}{z})$. Together with the $n(m-1)$ many zeros at points $[a_k:0:1]$, we have found $n(n-2)$ many zeros of it. Again, since $d(\frac{x}{z})$ has $n(n-3)$ more zeros than poles, these are all the zeros of $d(\frac{x}{z})$.

Next, just as before, the zeros and poles of the function $\frac{y}{z}$ are:
1 zero at each $[a_k:0:1]$, and n poles at $[0:1:0]$,
So the global holomorphic differential $ds=\varphi(y^{m-2}(c_{n-m-1}y^{n-m-1}+...+c_0z^{n-m-1}))$ exactly accounts for the difference of zeros and poles between $d(\frac{x}{z})$ and $\frac{y}{z}$, and the theorem follows. □

2 Initial value problems

Now we briefly consider the initial value problems of (1). Namely, the following:

$$(y')^{n-1}=g(y),\quad y(0)=a \tag{2}$$

We restrict our attention to the case where $g(a)\neq 0$. (Although it's entirely possible to also discuss the case when $g(a)=0$, it seems to deviate from the main line of this paper.) Also, let's first assume that all complex roots of $g(y)$ are simple roots.

First, for the complex case, we let θ denote a primitive $(n-1)$th root of unity, and we pick an arbitrary branch for the function $z \to z^{\frac{1}{n-1}}$ away from 0. Then, any local solution to (2) is a local solution to

$$y' = \theta^k g(y)^{\frac{1}{n-1}}, \quad y(0) = 0 \tag{3}$$

for some integer $0 \le k \le n-2$. But by the fundamental theorem of ODE, the local solution to (3) is unique for each k. So (2) has $n-1$ many local solutions.

On the other hand, let $y = y_0(x)$ be a local solution to (3) when $k = 0$, then $y = y_0(\theta^k x)$, $0 \le k \le n-2$ are all solutions to (2). So these are all the local complex solutions to (2).

Next we turn to the real case. i.e. when $g(y)$ is a real polynomial, and a is a real number, and we are interested in local real solutions. For appropriate choice of the branch of the function $z \to z^{\frac{1}{n-1}}$, (3) is a real problem and has a unique real local solution, which we can still denote by $y = y_0(x)$. Then (2) has at least one local real solution given by $y = y_0(x)$, and at most two solutions given by $y = y_0(x)$ and $y = y_0(-x)$ when $n-1$ is even. (It's possible that they are the same solution, for example think of the Weierstrass $\wp$ function.)

So, to find solutions to these initial value problems, we can first pick a point P on the curve C where $\frac{x}{z}$ attains the value a (P definitely exists as the nonconstant function $\frac{x}{z} - a$ on the curve has to have zeros). Then we may write down a solution to (2) according to theorem 2.1, which we denote by $y = y(x)$. then according to the above discussion, $y = y(\theta^k x)$ gives all local complex solutions to (2), and $y = y(\pm\theta^k x)$ gives all possible real solutions for some k, and they are locally real analytic.

For the case when $g(y)$ has roots of multiplicity greater than 1, we may simply deform $g(y)$ a little bit by adding some parameters and reduce the problem to the previous case. Then we take a limit in the solution for our parameters tending to zero. The validity of this is guaranteed by the continuous dependence of solutions to (2) on those parameters.

This completes our discussion on initial value problems of (1).

We end our discussion by an explicit algebraic geometric interpretation of solutions to $y' = y(y-b)$. Although this equation does not belong to the cases we have discussed explicitly so far (as we assumed $n \ge 3$ in section 2), however, this case is even simpler than what we have discussed because of its low degree. The corresponding curve $yz = x(x-bz)$ gives a 2-uple embedding of $\mathbb{CP}^1$ into $\mathbb{CP}^2$, where the function $\frac{x}{z}$ serves as a good meromorphic coordinate on this curve with only a simple pole at infinity, which we denote by s_1. According to the general method used in this paper, we need to find a local holomorphic coordinate s such that ds has two simple poles at $[0:0:1]$ and $[b:0:1]$ respectively. So $ds = \frac{ds_1}{s_1(s_1-b)}$ is a good choice.

When $b \ne 0$, this gives us $ds = \frac{1}{b} d \log \frac{s_1-b}{s_1}$, thus locally $s + c = \frac{1}{b} \log \frac{s_1-b}{s_1}$ for some constant c, so $\frac{x}{z} = s_1 = \frac{b}{1-e^{b(s+c)}}$ gives the expression of $\frac{x}{z}$ locally as a function of s_1, which is indeed the general solution to $y' = y(y-b)$.

When $b = 0$, we just take the limit of the above solution as $b \to 0$, then we get the general local solution to $y' = y^2$.

References

[1] S. Lang, *Elliptic Functions*, GTM 112

[2] R. Hartshorne, *Algebraic Geometry*, GTM 52