RESEARCH STATEMENT

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In the following I will describe four related continuing projects in mathematical physics that I am involved in: Nahm's conjecture and integrable quantum field theory in two dimensions, mathematical quantum field theory (following [Borcherds]), divergent series and the Virasoro algebra, reciprocity laws and physics. A central theme for my research is the relation between quantum field theory and number theory or algebra.

1. Nahm's conjecture and integrable quantum field theory in two dimensions

Many interesting mathematical problems arise from studying certain integrable quantum field theories in two dimensions. As is mentioned in [Nahm], some of these mathematical problems are very interesting yet look quite accessible such that one may expect to make progress. A particular example is Nahm's conjecture: we consider the q-hypergeometric series

$$f_{A,B,C}(z) = \sum_{n=(n_1,...,n_r)\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^tAn+B^tn+C}}{(q)_{n_1}...(q)_{n_r}}$$

where $(q)_n$ denotes the product $(1-q)(1-q^2)...(1-q^n)$, A is a positive definite symmetric $r \times r$ matrix, B is a vector of length r, and C is a scalar, all three with rational coefficients. Associated with $A = (a_{ij})$ we consider the system of r algebraic equations of r variables $x_1, ..., x_r$ (some additional care should be taken when A has nonintegral entries, see [Zagier])

$$1 - x_i = \prod_{j=1}^{r} x_j^{a_{ij}} \qquad (i = 1, ..., r) \tag{1.1}$$

We expect that there are only finitely many solutions to (1.1), so all solutions lie in $\overline{\mathbb{Q}}$. For any solution $x = (x_1, ..., x_r)$, we consider the element $\xi_x = [x_1] + ... + [x_r] \in \mathbb{Z}[F]$, where F is the number field $\mathbb{Q}(x_1, ..., x_r)$. ξ_x defines an element in the Bloch group $\mathcal{B}(F)$ (see [Zagier]). Then Nahm's conjecture asserts that the following are equivalent:
(i) The element ξ_x is a torsion element of $\mathcal{B}(F)$ for every solution x of (1.1)
(ii) There exist B and C such that $f_{A,B,C}(z)$ is a modular function.

Nahm's conjecture provides a deep relation between the modularity of a class of q-hypergeometric series and torsion elements in the Bloch group of certain number fields, which are closely related to the third algebraic K group of these number fields. (1.1) has been studied by various people when the matrix A can be written as a

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Kronecker product of a Cartan matrix and the inverse of another one, both of ADET type. In this case representation theory of Lie algebras and quantum groups fits into the picture and produces solutions, and people obtained interesting dilogarithm identities as consequences. One may consult [Kirillov], [KR] for examples. Also, the conditions (i), (ii) above are expected to both hold in this case.

This conjecture is proven in rank 1 case, but is otherwise wide open. Excellent numerical evidence is found in [Zagier]. Obviously, it will be very useful if one can get a complete list of matrices A such that condition (i) above holds (Let us denote this list by L). Presumably, however, this is a hard task.

My idea is to first ask an easier question: to determine the matrices A such that all solutions to (1.1) are real. For any number field F, it is well known that the free part of $\mathcal{B}(F)$ is isomorphic to $\mathbb{Z}^{r_2}$, where $2r_2$ is the number of complex embeddings of F. Therefore for each rank r, the set of these matrices is a proper subset of L, since they give rise to totally real number fields, for which any Bloch group element is automatically torsion. We wish to investigate this subset for two reasons: first, it is reasonable to expect that this subset constitutes a substantial part of L (From [Zagier] and [Nahm], one may look at the available examples of Nahm's conjecture to get an idea of this. Except some trivial infinite families, most examples are in this subset.); second, we hope this subset is much more tractable.

In [HL2], Chul-hee Lee and I proved the following

Theorem 1.1. *If* $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ *is a positive definite symmetric matrix with integer entries such that all complex solutions to the system of equations*

$$1 - x_1 = x_1^a x_2^b$$
$$1 - x_2 = x_1^b x_2^d$$

are real, then $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ *equals one of* $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

This result partially realizes the above idea for attacking Nahm's conjecture for rank 2 case. Furthermore, it looks likely that our method can be generalized to A with rational entries, and also to higher rank cases. The former looks straightforward but probably messy, and the latter looks quite interesting as the theory of fewnomials naturally comes into the picture, and one needs to combine this with an appropriate generalization of Bezout's theorem to get the desired generalization. I am interested in trying to provide such generalizations in the near future. It looks that this line of development is somewhat parallel to the older idea described above involving Cartan matrices and representation theory.

On the other hand, as is said in [Nahm], it is important and much more challenging to develop a mathematical framework for integrable quantum field theories in two dimensions, and one may start with integrable perturbations of rational conformal field theories since they are probably the easiest ones to study. To have a glimpse of what possible mathematical mystery are there, one may take a look at some strikingly curious q-series identities such as equation (5) in [Nahm]. The mathematics of Nahm's conjecture, or more generally of integrable quantum field theories in two dimensions, should be very rich yet accessible and lies at a crossroad of representation theory, modular forms, algebraic K-theory, and algebraic geometry. Exploring these is one of my long-term goals.

2. Mathematical quantum field theory

2.1. **Background.** While using the same word quantum field theory, most of the time, mathematicians and physicists are doing very different things. I keep a keen interest in searching for a mathematically rigorous framework of quantum field theory in 4 dimensions, which can in principle produce scattering amplitudes observed in experiments. In my opinion, the article [Borcherds] is an important step forward in this direction. It captures important and essential mathematical contents in physics textbooks, especially renormalization, in a rigorous way. On one hand, physical ideas can be very powerful, and can bring mathematical conjectures beyond imagination of mathematicians. On the other hand, it often happens that mathematical solutions can change the physical stories, making them much more elegant and clearer. This philosophy is realized in a lot of articles such as [Borcherds].

2.2. Anomalies are treated rigorously as elements in certain (noncommutative) group cohomology, which have been calculated in a lot of cases in [BBH]. With the same philosophy, in the technical paper [Huang4], I simplified some crucial calculations of [BBH] by using spectral sequence of a double complex, which can hopefully make *loc cit* easier to access by mathematicians. With this specific goal in mind, I tried to make the paper as short as I can, while explaining things clearly.

Furthermore, [Borcherds] proved a general theorem about the existence of perturbative Feynman measures, and renormalization is revealed to have very intricate algebraic structures, where Hopf algebras and Bernstein-Sato polynomials play crucial roles. There is a fascinating conjectural relationship between knots and multiple zeta values as revealed by Kreimer and Broadhurst, who discovered them when exploring algebraic structures in renormalization theory of perturbative quantum field theory. As we are becoming able to treat renormalization rigorously, I think this conjecture is becoming more accessible by mathematicians. I am interested to understand the mathematics underlying these conjectural relations.

I also plan to hold seminars discussing and possibly extending [Borcherds], which will be based on his article and his Fall 2010 course on quantum field theory at Berkeley.

3. Divergent series and the Virasoro algebra

3.1. Background. In bosonic string theory we have a famous equality $0+1+2+3+...=-\frac{1}{12}$. This is closely related to the fractional power $\frac{1}{24}$ of q in the Dedekind η function

$$q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

Roughly speaking, the fractional power of q here is physically explained as the ground state Casimir energy of a (holomorphic) conformal field theory by summing up all zero-point energies, and the modularity of the Dedekind η function is related to the zero-point correlation function of this conformal field theory on a torus. One can make this story formally rigorous by the theory of vertex operator algebras. In particular there is a fundamental paper [Zhu] on the modularity of certain conformal field theory partition functions, and the work of Spencer Bloch [Bloch] on divergent series and special zeta or L values, and later works of Anton Milas, James Lepowsky and others based on Bloch's ideas.

3.2. In [Huang1] I rediscovered the "twisted Virasoro operators" first considered by Spencer Bloch in 1996 [Bloch]. (I became aware of Bloch's work after I submitted my paper to `arxiv.org`.) Furthermore I used this to obtain an explicit formula expressing the rational powers of q in product expansions of certain modular forms as a linear combination of special Dirichlet L values at -1:

$$h^{1,j}_{2,2k+1} - \frac{c_{2,2k+1}}{24} = \frac{1}{2}L(-1,\chi^k) - \frac{1}{2k}\sum_{s=1}^{k}\omega^{is}L(-1,\chi^s) \tag{3.1}$$

Where $2k+1$ is an odd prime, and $h^{1,j}_{2,2k+1} - \frac{c_{2,2k+1}}{24}$ is the fractional power of q appearing in the product expansion of the character of the $(2,2k+1)$ minimal model with highest weight $h^{1,j}_{2,2k+1}$:

$$\overline{ch}_{c_{2,2k+1},h^{1,j}_{2,2k+1}}(q) = q^{h^{1,j}_{2,2k+1}-\frac{c_{2,2k+1}}{24}} \prod_{n\neq\pm j,0 \mod 2k+1} \frac{1}{(1-q^n)}$$

ω is any primitive kth root of unity, and χ is a chosen generator of the cyclic group of even Dirichlet characters of conductor $2k+1$ so that $j, 2k+1-j$ is the unique pair such that

$$\chi(j) = \omega^{k-i}$$

In [Huang1] we actually get an equality like (3.1) in more general cases, e.g. when $2k+1$ is not necessarily prime. Furthermore these Dirichlet characters give rise to the number field $\mathbb{Q}(\cos\frac{2\pi}{2k+1})$, which coincides with the number field we get from solving Nahm's algebraic equations for $2T^{-1}$ (The matrix $2T^{-1}$ corresponds to $(2,2k+1)$ model in a specific way, serving as one of the first examples of Nahm's conjecture, and corresponds also to Andrews-Gordan identities.), where T is the Cartan matrix given by the tadpole Dynkin diagram of rank $k-1$. I am currently trying to understand this coincidence and related topics, which sits at a crossroad of conformal field theory, Dilogarithm function, hyperbolic volumes, etc.

Also, I studied certain axioms for summing up divergent series. I explored the relations of these axioms with those 'twisted Virasoro operators', and special Dirichlet L values at 0 and 1. In particular I interpreted the sum $0+1+2+... = -\frac{1}{12}$ from an axiomatic point of view related with a scaling symmetry of the conformal field theory of free massless bosons. As a byproduct I obtained a way of calculating special Dirichlet L values $L(0,\chi)$ and $L(-1,\chi)$ by certain axioms of divergent series. I am interested in studying these divergent series further.

4. Reciprocity laws and physics

The relations between geometric Langlands duality and electric-magnetic duality of gauge theories are being extensively studied. There are a lot of conferences and workshops on this topic, one can for example go to the webpage of Edward Frenkel to get some details. There is work on defining a quantum field theory on an algebraic curve, and physical interpretations of reciprocity laws. One may see [Witten], or [Takhtajan]. I am currently trying to make sense of a quantum field theory on a number field, and then give physical interpretations of number theory reciprocity laws by dualities of physics.

My paper [Huang3] is a speculative work on this topic. I discussed some ideas of formulating some quantum field theories on number fields (with certain symmetries mimicking conformal symmetry or gauge symmetry of ordinary quantum field theory), and then focused on a possible concrete physical interpretation of the Gauss reciprocity law in terms of a duality of quantum field theory. Along the way, many subtle ingredients of number theory and quantum field theory get closely related, such as the class number and the dimension of the space of conformal blocks. The analogy between knot theory and number theory also plays an important role. I hope to continue on this project by collaborating with other researchers, refining my paper [Huang3], and developing the ideas. This is a long-term project, but I am expecting some interesting results in the near future, such as a well-defined dictionary between some concepts in number theory and quantum field theory on number fields.

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