Math 53, Spring 2000, sections 107 & 109

Review Sheet

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Here are the problems which were submitted for extra credit (some of them have been slightly edited):

1. (Tommy Sy) Find the area of the portion of the plane $3x+2y+6z=6$ which is bounded by the coordinate planes.

2. (Tommy Sy) Evaluate the integral

$$\iint_R \frac{x-y}{x+y}\,dA$$

where R is the square with vertices $(0,2)$, $(1,1)$, $(2,2)$, and $(1,3)$. **Hint**: Use the transformation $u=x-y$, $v=x+y$.

3. (Eric Ng) The plane $x+y+2z=2$ intersects the paraboloid $z=x^2+y^2$ in an ellipse. Find the points on this ellipse that are closest to and furthest from the origin.

4. (Eric Ng) If $z=f(u,v)$ where $u=xy$, $v=\frac{y}{x}$, and f has continuous second derivatives, show that

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=-4uv\frac{\partial^2 z}{\partial u\,\partial v}+2v\frac{\partial z}{\partial v}.$$

5. (Tuji Chang) Find $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z)=2xy\,\mathbf{i}+xz\,\mathbf{j}+yz\,\mathbf{k}$ and S consists of the following three pieces, with outward orientation:

$$x^2+y^2=1, 0\le x, 0\le z\le 2,$$
$$z=0, 0\le x, x^2+y^2\le 1,$$
$$z=2, 0\le x, x^2+y^2\le 1.$$

6. (Tuji Chang) Evaluate the line integral $\int_C y\,dx+z\,dy+x\,dz$, where C consists of two line segments from $(0,0,0)$ to $(1,1,2)$ and from $(1,1,2)$ to $(3,1,4)$.

7. (Allison Ryan) Prove that the maximum volume of a box with fixed surface area is achieved when all side lengths are equal.

8. (Allison Ryan) Find the mass of a sphere of radius a whose density (in the appropriate units) is equal to $2r^2$, where r is the distance from the origin.

9. (Allison Ryan) Find the flux of the vector field $\mathbf{F}(x,y,z)=\langle x,y^2,2x+5\rangle$ through the cylinder $x^2+y^2=1, -1\le z\le 1$, oriented outward.

10. (Bobby Young) Evaluate $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z)=xyz\,\mathbf{i}+xy\,\mathbf{j}+x^2yz\,\mathbf{k}$ and S is the top and four sides (but not the bottom) of the cube with vertices $(\pm1,\pm1,\pm1)$, oriented outward.

11. (Bobby Young) Find the area of the surface defined by the parametric equations $x = uv, y = u + v, z = u - v$ over the region $u^2 + v^2 \leq 1$.

12. (Bobby Young) Show that every plane that is tangent to the cone $x^2+y^2 = z^2$ passes through the origin.

13. (Albert Lee & Steve Hong) Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 2$.

14. (Albert Lee & Steve Hong) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and C is the intersection of the surfaces $z = x^2 + y^2$ and $x + 2y + z = 6$, oriented counter-clockwise as viewed from above.

15. (Albert Lee & Steve Hong) Find the (shortest) distance between the following two skew lines:

$$\begin{array}{llll} \ell_1 : & x = 2 + t & y = 5 - t & z = 3t \\ \ell_2 : & x = 4 - s & y = 6s & z = 7s + 1. \end{array}$$

16. (Matthew Iong) If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are constant vectors, $\mathbf{r}$ is the position vector $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and E is the region defined by the inequalities $0 \leq \mathbf{a} \cdot \mathbf{r} \leq \alpha$, $0 \leq \mathbf{b} \cdot \mathbf{r} \leq \beta$, and $0 \leq \mathbf{c} \cdot \mathbf{r} \leq \gamma$, show that

$$\iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r})\, dV = \frac{(\alpha\beta\gamma)^2}{8|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}.$$

17. (Matthew Iong) Evaluate $\int_C (y + \sin x)\, dx + (z^2 + \cos y)\, dy + x^3\, dz$, where C is the curve $\mathbf{r}(t) =< \sin t, \cos t, \sin 2t >$, $0 \leq t \leq 2\pi$.

Hint: C lies on the surface $z = 2xy$.

18. (Matthew Iong)

a. Evaluate $\iint_D \frac{1}{(x^2+y^2)^{\frac{n}{2}}}\, dA$, where n is an integer and D is the region between circles centered at the origin of radius r and R, $0 \leq r \leq R$.

b. For what values of n does the integral in part (a) have a limit as $r \to 0^+$?

c. Find $\iiint_E \frac{1}{(x^2+y^2+z^2)^{\frac{n}{2}}}\, dV$, where n is an integer and E is the region between spheres centered at the origin of radius r and R, $0 \leq r \leq R$.

d. For what values of n does the integral in part (c) have a limit as $r \to 0^+$?

19. (Yasemin Salavatcioglu) Find the volume of the region bounded by these equations:

$$\begin{aligned} 3x + 4y - 2z &= 0 \\ x &= 2y \\ x &= 2 \\ z &= 0. \end{aligned}$$

20. (Yasemin Salavatcioglu) Find the area of the surface $4x^2 + 4y^2 + z = 1$ which lies above the (x, y)-plane.

21. (Easan Drury) Evaluate the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$ and S is that part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, and S is oriented upward.

22. (Easan Drury) Evaluate $\iint_R (x^2 - xy + y^2)\, dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$.

Hint: Find a change of variables which transforms the boundary to an equation of the form $u^2 + v^2 = 1$.

23. (Greg Beck) Find the volume of the region bounded by the surfaces $x^2 - 6x + y^2 - 4y - z = -13$ and $z = 4$.

24. (Amy Platt) Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the portion of the surface $x^2 + y^2 - z^2 = 1$ which lies inside the cylinder $x^2 + y^2 = 4$, oriented outward, and $\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$.

25. (Amy Platt) Find the minimum and maximum values of the function $f(x, y) = 3x^2 - 2y^2$ over the region $x^4 + y^4 \le 2$.

26. (Amy Platt) Find the volume of the tetrahedron bounded by the equations $\frac{z}{3} + \frac{y}{2} + x = 1$, $z = 0$, $x = 0$, and $x = y$.

27. (Amy Platt) Evaluate $\iint_S xy\, dS$, where S is the surface $z = x^2 y - y^2 x$ over the region $0 \le x \le 2$, $0 \le y \le 3$.

28. (James Junio) Find $\iint_S f\, dS$ if $f(x, y, z) = \sqrt{xy^2 + xz^2}$ and S is the cone $x^2 = z^2 + y^2$, $0 \le x \le 2$.

29. (James Junio) Find $\int_C y\, dx + z\, dy + x\, dz$ if C is the line segment which runs from the point $P = (2, 2, 3)$ to the point on the plane $x + 2y + 3z = 1$ which is closest to point P.

30. (Damian Sowinski) Find the mass of the solid inside the sphere $x^2 + y^2 + z^2 = a^2$ but outside the cylinder $x^2 + y^2 = \left(\frac{a}{2}\right)^2$, whose density is given by $\rho(x, y, z) = \sqrt{a^2 - x^2 - y^2 - z^2}$.

31. (Damian Sowinski) Describe the set of level surfaces of the function $f(x, y, z) = x^2 - y^2 + z$.

32. (Hank Fung) Find the maximum and minimum values of the function $f(x, y, z) = 3x - y - 3z$ subject to the constraints $x + y - z = 0$ and $x^2 + 2z^2 = 1$.

33. (Hank Fung) Evaluate the surface integral $\iint_S (x^2 z + y^2 z)\, dS$ where S is that part of the plane $z = 4 + x + y$ which lies inside the cylinder $x^2 + y^2 = 4$.

34. (Hank Fung) Evaluate the triple integral $\iiint_E z\, dV$ where E is the region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $z = 0$, and $y = 3x$.

35. (Chris Wahl) Find the flux of $\mathbf{F}$ through the surface S, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$ and S is the surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

36. (Chris Wahl) Determine whether or not the vector field $\mathbf{F}(x, y) = (x + y^2)\,\mathbf{i} + (2xy + y^2)\,\mathbf{j}$ is conservative. If it is, find a function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

37. (Scott Heise) Use Stoke's Theorem to evaluate $\int_C \mathbf{F} \cdot \mathbf{r}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + 2xy\,\mathbf{j} + 3xy\,\mathbf{k}$ and C is composed of three line segments traversing the points $(1, 0, 0)$, $(0, 3, 0)$, $(0, 0, 3)$, and $(1, 0, 0)$ again in order.

38. (Scott Heise) Use Stoke's Theorem to evaluate $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$ and S is the portion of the sphere $x^2+y^2+z^2=1$ above the plane $z=\frac{\sqrt{2}}{2}$.

39. (Scott Heise) Evaluate the surface integral $\iint_S xyz\,dS$, where S is the portion of the sphere $x^2+y^2+z^2=1$ which lies above the cone $z=\sqrt{x^2+y^2}$.

40. (Laura Gambs) A silo in the shape of a cylinder with radius r sits in a field of grass. At one point on the outside of the silo, a cow is tethered by a rope just long enough to wrap halfway around the silo, allowing the cow to reach the opposite side. What is the total amount of area in which the cow can graze?

41. (Laura Gambs) In a few sentences describe all of the major theorems of Chapter 14, and how they relate to each other. Include a physical interpretation of each equation.

42. (Mujtaba Saifuddin) Find the equations of (a) the tangent plane and (b) the normal line to the surface $x^2-2y^2+z^2=3$ at the point $(-1,1,-2)$.

43. (Mujtaba Saifuddin) Find a parametric representation for the portion of the elliptic paraboloid $x+y^2+2z^2=4$ with $x\geq 0$.

44. (Nicole Foletta) Calculate the work done by the force field $\mathbf{F}(x,y,z) = (x^x+z^2)\,\mathbf{i} + (y^y+x^2)\,\mathbf{j} + (z^z+y^2)\,\mathbf{k}$ when a particle moves under its influence around the boundary of that part of the surface $x^2+y^2+z^2=4$ which lies in the first octant, in a counterclockwise direction as viewed from above.

45. (Nicole Foletta) Verify by direct calculation that Stoke's Theorem is true when applied to the vector field $\mathbf{F}(x,y,z) = y\,\mathbf{i}+z\,\mathbf{j}+x\,\mathbf{k}$ and the surface $x+y+z=1$ in the first octant, oriented upward.

Here are a few additional problems that I came up with, sometimes based on the problems above (especially difficult problems are marked with an asterisk):

A. Find the center of mass of the surface described in Problem 1, if the density is proportional to the distance from the origin.

B. *Find the flux of the vector field

$$\mathbf{F}(x,y,z) = \frac{(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

through the surface in Problem 1, with upward orientation. (If you're curious, the flux of this particular vector field through a surface measures the amount of light hitting it from a point source at the origin.)

Hint: In order to get an integral you have some hope of evaluating, calculate the flux through an octant of a sphere together with pieces of the three coordinate planes, and then use the divergence theorem to prove that the two integrals are equal.

C. Calculate the flux of the vector field from Problem B through the unit sphere, oriented outward. Why doesn't this contradict the divergence theorem? (Here's a related question: Why isn't there really such a thing as a "point source" of light?)

D. Evaluate the limit $\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x+y}$, or demonstrate that the limit does not exist.

Hint: The limit doesn't exist. Saying "the limit is $\frac{0}{0}$" or "the function is undefined along the line $y = -x$" is NOT enough to show that the limit doesn't exist.

E. What are the best angles of boat and sail (two different angles) for "tacking" straight into the wind? (The force produced by the wind on the sail is perpendicular to the sail, with a magnitude proportional to the flux of the wind velocity field through the sail. The motion of the boat is determined by projecting the force vector onto a vector perpedicular to the keel of the boat, and the amount of forward progress of the boat against the wind is obtained by projecting the boat's velocity onto a vector in the opposite direction of the wind. How many sailors realize that sailing against the wind is only possible because of the mathematical curiousity that the direction of a vector may be reversed by applying a series of three projections?)

F. If $\langle x, y, z\rangle = \mathbf{r}(u, v)$ is a parametrized surface, then the tangent plane at the point $\mathbf{A} = \langle a, b, c\rangle$ can be parametrized as $\langle x, y, z\rangle = \mathbf{A} + s\mathbf{r}_u + t\mathbf{r}_v$. Explain why. Expand this vector equation into three scalar equations and eliminate s and t to get a single equation in x, y, and z. Read the normal vector to the plane off of the coefficients of your equation, and verify that it is a multiple of the vector $\mathbf{r}_u \times \mathbf{r}_v$.

G. Draw a picture of Stoke's Theorem in action. Explain it to someone who is not in Math 53.

H. Draw a picture of the Divergence Theorem in action. Explain it to someone who is not in Math 53.

I. Prove that Stoke's Theorem applied to a surface in the (x, y)-plane is equivalent to Green's Theorem.

J. What is the two-dimensional analogue of the Divergence Theorem? Show that this is equivalent to Green's Theorem.

K. What is the one-dimensional analogue of the Divergence Theorem? (This is also equivalent to some well-known theorem . . .)

L. Use Stoke's Theorem to prove that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ is zero for any closed surface S.

M. Use the Divergence Theorem to prove that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ is zero for any closed surface S.