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Math 53
5/5/00

Practice Final

Problem 1: (a) Suppose that we have a function $f(r,\theta)$, where r and θ are the usual polar coordinates. Show that

$$\nabla f(r,\theta) = (\frac{x}{r}f_r - \frac{\sin\theta}{r}f_\theta, \frac{y}{r}f_r + \frac{\cos\theta}{r}f_\theta).$$

(b) Verify that this formula holds for $f(r,\theta) = r^n$.

Problem 2: (a) Write the equation for the tangent plane to the surface $f(x,y,z) = 0$ at a point (x_0, y_0, z_0) on the surface.

(b) Let $\mathbf{u}$ be a vector parallel to this plane. What can you say about $D_{\mathbf{u}} f(x_0, y_0, z_0)$?

Problem 3: Find the points on the sphere of radius 2 centered at the origin which are closest to and farthest from the point $(3, 1, -1)$.

Problem 4: Compute $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$.

Problem 5: The center of mass of a solid with constant density is called the centroid of the solid. Find the volume and centroid of the solid bounded by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 1$.

Problem 6: Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Suggestion: change variables so that the ellipsoid is a sphere. Check that your answer works when $a = b = c$.

Problem 7: Let C be the curve $\{(\cos t, t, \sin t) : 0 \le t \le \pi\}$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where

$$\mathbf{F}(x,y,z) = (3x^2\sin y + ze^z)\mathbf{i} + x^3\cos y\mathbf{j} + (z+1)xe^z\mathbf{k}.$$

Problem 8: (a) Define $\nabla^2 h = \nabla\cdot(\nabla h)$, for any twice differentiable function h. Show that if f and g are two twice differentiable functions, then

$$\nabla\cdot(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g.$$

(b) Show that in addition, if E is a region bounded by a surface S which is oriented outward, then

$$\iint_S (f\nabla g - g\nabla f)\cdot d\mathbf{S} = \iiint_E (f\nabla^2 g - g\nabla^2 f)\, dV.$$

Problem 9: Let C be the curve obtained by intersecting the plane $x+2y+z = 4$ with the cylinder $x^2+y^2 = 4$. Orient C counterclockwise as viewed from above. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = z\mathbf{i} - x\mathbf{j} + y\mathbf{k}$.

Problem 10: Given a parallelogram in the plane, let a and b be the lengths of the sides, and let c and d be the lengths of the diagonals. Show that $c^2+d^2 = 2a^2+2b^2$.