Math 32, Fall 2001, Tracy Hall
Solutions to Midterm II, 4 May
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1a) Fill in the blanks (2 points $\times$ 10 blanks = 20 points):

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$
30^o	$\frac{\pi}{\mathbf{6}}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}$
$\mathbf{45^o}$	$\frac{\pi}{4}$	$\frac{\sqrt{\mathbf{2}}}{\mathbf{2}}$	$\frac{\sqrt{\mathbf{2}}}{\mathbf{2}}$
240^o	$\frac{\mathbf{4}\pi}{\mathbf{3}}$	$\frac{-\sqrt{\mathbf{3}}}{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{2}}$
$0 < \theta < 90^o$	$0 < \theta < \frac{\pi}{2}$	$\frac{3}{5}$	$\frac{\mathbf{4}}{\mathbf{5}}$

1b) (5 points) Simplify the following expression:

$$\sin^2(\pi - 2\beta) + \cos^2(\pi - 2\beta)$$

$$\sin^2(\theta) + \cos^2(\theta) = \mathbf{1},$$

even when θ is something like $\pi - 2\beta$.

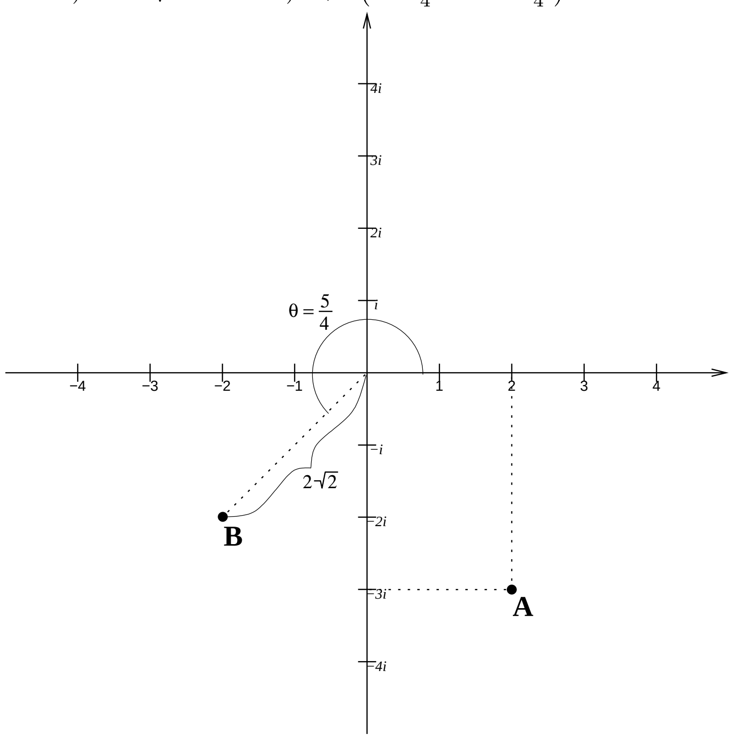
2a) (5 points) What are the amplitude and period of the function

$$y = 2\cos\left(\frac{\pi}{3}x - \pi\right)?$$

Amplitude: 2

Period: $\mathbf{2\pi/\frac{\pi}{3} = \frac{2\pi}{1}\frac{3}{\pi} = 6}$

2b) (4+6 points) Plot and label the following complex numbers in the complex plane: A) $2 - 3\sqrt{-1}$ and B) $2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$.



2c) (5+5 points) Compute the product of the two complex numbers

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

in **two different ways**: both by multiplying it out, and by addition of angles. (This is how we derived the angle addition formulas for cosine and sine.)

Method 1: multiply it out

$$\begin{aligned} & \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta \\ = \; & [\cos\alpha\cos\beta - \sin\alpha\sin\beta] + i\,[\cos\alpha\sin\beta + \sin\alpha\cos\beta] \end{aligned}$$

Method 2: addition of angles

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

3a) (4 points) Add the fractions $\frac{\pi}{3} + \frac{-\pi}{4}$ and simplify.

$$\frac{4\pi}{12} + \frac{-3\pi}{12} = \frac{\pi}{12}$$

3b) (8 points) Compute $\cos\frac{\pi}{12}$ and simplify your answer. (The previous question was a hint.)

$$\begin{aligned} & \cos\left(\frac{\pi}{3} + \frac{-\pi}{4}\right) \\ = \; & \cos\frac{\pi}{3}\cos\frac{-\pi}{4} - \sin\frac{\pi}{3}\sin\frac{-\pi}{4} \\ = \; & \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{-\sqrt{2}}{2} \\ = \; & \frac{\mathbf{\sqrt{2} + \sqrt{6}}}{\mathbf{4}} \end{aligned}$$

3c) (8 points) Compute $\sin\frac{\pi}{12}$ and simplify your answer.

$$\begin{aligned} & \sin\left(\frac{\pi}{3} + \frac{-\pi}{4}\right) \\ = \; & \cos\frac{\pi}{3}\sin\frac{-\pi}{4} + \sin\frac{\pi}{3}\cos\frac{-\pi}{4} \\ = \; & \frac{1}{2}\frac{-\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\ = \; & \frac{\mathbf{-\sqrt{2} + \sqrt{6}}}{\mathbf{4}} \end{aligned}$$

3d) (5 points) Now use the double-angle formula for sine to compute $\sin(2\frac{\pi}{12})$. Simplify your answer. (If your answer doesn't agree with the value for $\sin\frac{\pi}{6}$, then you've made a mistake somewhere on this page.)

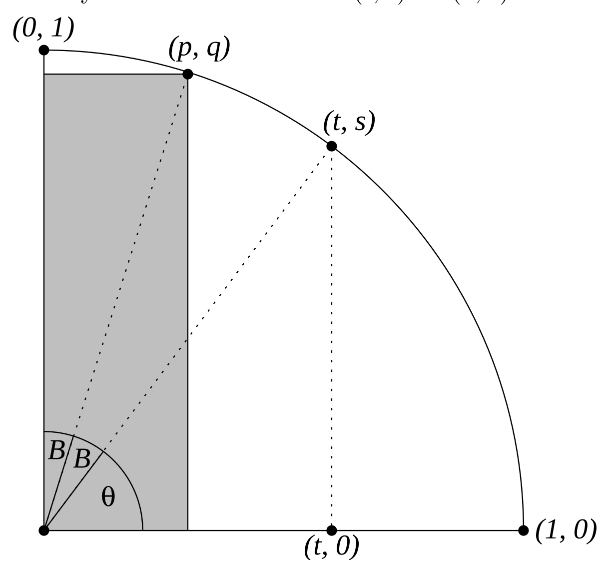
$$\begin{aligned} \sin\left(2\frac{\pi}{12}\right) &= 2\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right) \\ &= 2\left(\frac{-\sqrt{2}+\sqrt{6}}{4}\right)\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right) \\ &= 2\frac{-2-\sqrt{12}+\sqrt{12}+6}{16} \\ &= \frac{2\cdot 4}{16} = \frac{\mathbf{1}}{\mathbf{2}}. \end{aligned}$$

4a) (6 points) Write the angle addition formulas for $\cos(A+B)$ and $\cos(A-B)$. Add the two equations and obtain a product-to-sum formula.

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ \cos(A-B) &= \cos A\cos B + \sin A\sin B\\ \cos(A+B)+\cos(A-B) &= 2\cos A\cos B,\ \text{and so}\end{aligned}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right].$$

The figure below shows the first quadrant of the unit circle in the (x, y)-plane, with various points and angles labeled. Notice that (t, s) is directly above the point $(t, 0)$, and that (p, q) is exactly halfway around the arc from (t, s) to $(0, 1)$.



4b) (9 points) Fill in the missing angles:

$$t = \cos(\ \theta\ ) \qquad p = \cos(\theta + B) \qquad q = \cos(\ B\ )$$

4c) (10 points) What is the area of the shaded rectangle in terms of t? You should simplify your answer to get rid of any angles or variables other than t.

Hint: Use your answers to 4a) and 4b).

Multiplying the height of the rectangle by its width gives us

$$\begin{aligned}pq &= \cos(\theta+B)\cos(B)\\ &= \frac{1}{2}\left[\cos(\theta+B+B)+\cos(\theta+B-B)\right]\\ &= \frac{1}{2}\left[\cos(\frac{\pi}{2})+\cos(\theta)\right]\\ &= \frac{1}{2}\left[0+t\right] = \mathbf{\frac{t}{2}}\end{aligned}$$