Math 32, Fall 2001, Tracy Hall
Solutions to Midterm I, 6 March

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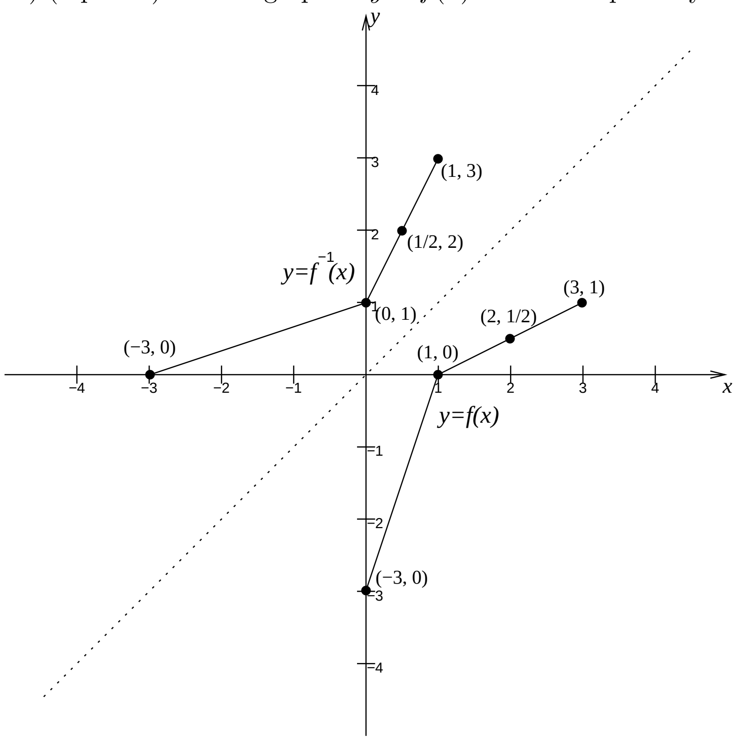
For all the problems on this page, let $f(x)$ be defined as follows:

$$f(x) = \begin{cases} 3x - 3, & 0 \le x \le 1 \\ \frac{1}{2}(x-1), & 1 < x \le 3 \end{cases}$$

1a) (5 points) Compute $f(0)$, $f(1)$, $f(2)$, and $f(3)$.
$f(0) = 3(0) - 3 = -3$
$f(1) = 3(1) - 3 = 0$
$f(2) = \frac{1}{2}(2-1) = \frac{1}{2}$
$f(3) = \frac{1}{2}(3-1) = 1$
Remember that a function can have only one output.

1b) (5 points) Draw a graph of $y = f(x)$. Label the points you computed.



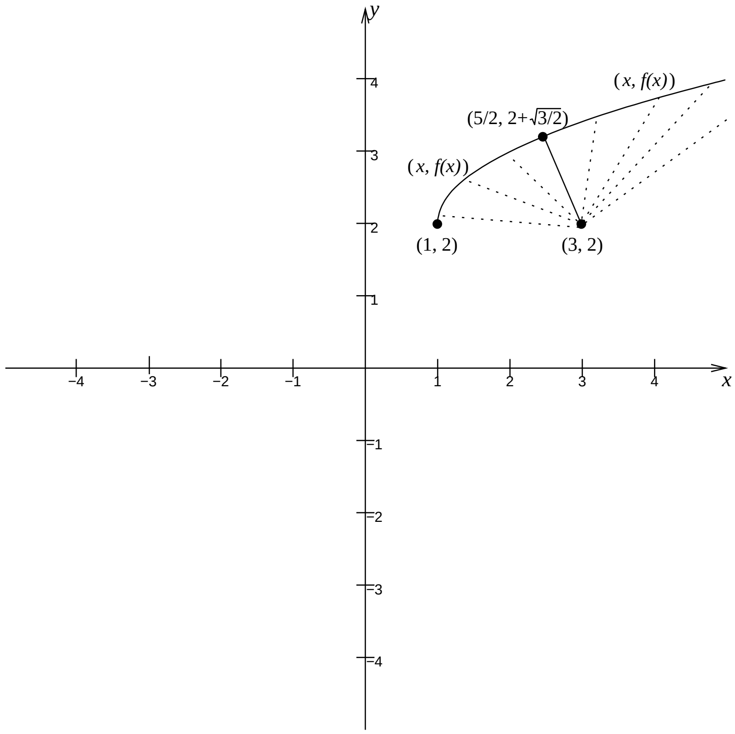
1c) (5 points) On the same axes, draw the graph of $y = f^{-1}(x)$.

1d) (10 points) Fill in the blanks:

$$f^{-1}(x) = \begin{cases} \mathbf{\frac{1}{3}}x + \mathbf{1}, & \mathbf{-3} \le x \le \mathbf{0} \\ \mathbf{2}x + \mathbf{1}, & \mathbf{0} < x \le \mathbf{1} \end{cases}$$

For all the problems on this page, let $f(x) = \sqrt{x-1} + 2$.

2a) (5 points) Draw the graph of $y = f(x)$.



2b) (10 points) Plot the point $(3, 2)$ on the same set of axes. Give the distance from $(x, f(x))$ to the point (3,2), as a function of x.

A: The distance is $\sqrt{(x-3)^2 + (f(x)-2)^2}$

$$= \sqrt{(x-3)^2 + \left(\sqrt{x-1} + 2 - 2\right)^2}$$

$$= \sqrt{(x-3)^2 + x - 1}$$

$$= \sqrt{x^2 - 5x + 8}.$$

2c) (10 points) Find the point on the curve $y = f(x)$ which is closest to the point $(3, 2)$.

A: We just need to find the x value which minimizes the square of the distance, or in other words, the x-coordinate of the vertex of the parabola $x^2 - 5x - 8$. We could complete the square, or just use the formula $\frac{-b}{2a}$, which gives us $x = \frac{5}{2}$.

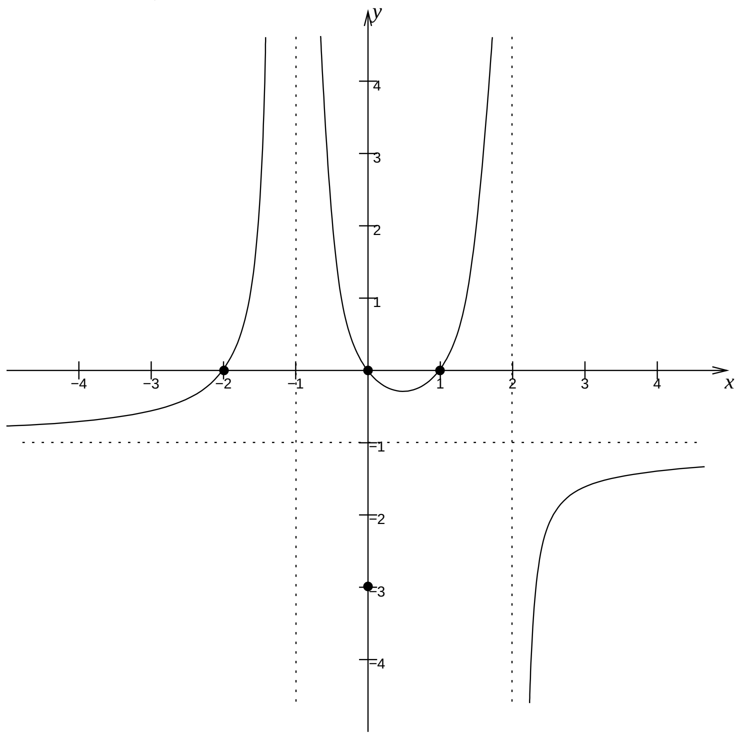
Now that we have the x-coordinate of the closest point, we need to plug that into the *original* function to find the y-value of the closest point:

$$f\left(\frac{5}{2}\right) = \sqrt{\frac{5}{2} - 1} + 2 = \sqrt{\frac{3}{2}} + 2.$$

For all the problems on this page, let $f(x)$ be the rational function

$$f(x) = \frac{-x(x+2)(x-1)}{(x+1)^2(x-2)}.$$

3a) (15 points) Draw the graph of the equation $y = f(x)$. Include any vertical or horizontal asymptotes.



3b) (5 points) Does the function $f(x)$ have an inverse? Explain why or why not.

A: No, $f(x)$ does not have an inverse. It fails the horizontal line test. For example, $f(0) = f(1)$.

3c) (5 points) Let $g(x)$ be the function $g(x) = \sqrt{x}$. What is the domain of the function $(g \circ f)(x)$? Express your answer in interval notation.

A: The function $(g \circ f)(x)$ is $\sqrt{f(x)}$, which expands to

$$\sqrt{\frac{-x(x+2)(x-1)}{(x+1)^2(x-2)}}.$$

The domain of this is all numbers for which $f(x) > 0$. We've already done all the work; we just need to refer to the graph of $f(x)$ and see when it is defined and positive.

$$\text{Domain: } [-2,-1) \cup (-1,0] \cup [1,2)\,.$$

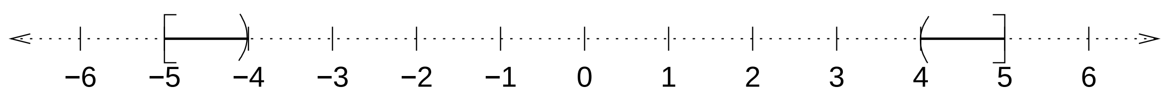
4a) (5 points) Simplify the following expression as much as possible, using the change-of-base formula and other properties of logarithms. (The simplified answer has only one logarithm in it.)

$$\begin{aligned} & \ln(\log_5 e^3) + \ln(\log_2 5) + \ln(\ln(2)) \\ = \quad & \ln\left(\frac{\ln e^3}{\ln 5}\right) + \ln\left(\frac{\ln 5}{\ln 2}\right) + \ln\left(\ln(2)\right) \\ = \quad & \ln\left(\frac{\ln e^3}{\ln 5}\frac{\ln 5}{\ln 2}\ln(2)\right) \\ = \quad & \ln\left(\ln e^3\right) \\ = \quad & \ln(3) \end{aligned}$$

4b) (10 points) Which values of x make the following inequality true? Graph your answer on the number line below. (Be careful not to include anything which is not in the domain of the function.)

$$\begin{aligned} \log_3(x^2 - 16) &\leq 2 \\ (x^2 - 16) &\leq 3^2 = 9 \\ x^2 &\leq 25 \end{aligned}$$

The key values for the inequality are $x = 5$ and $x = -5$ and the key values for the domain ($x^2 - 16 > 0$) are $x = 4$ and $x = -4$. By testing which numbers in the domain make the inequality true, we can fill in the number line as follows:



4c) (10 points + 5 point bonus) Solve for y in terms of x:

$$\begin{aligned} e^x &= e^y + x + 1 \\ -e^y &= -e^x + x + 1 \\ e^y &= e^x - x - 1 \\ y &= \ln(e^x - x - 1) \end{aligned}$$

This cannot be simplified further.

Bonus: What is the domain of your answer?

A: The domain is all numbers such that $e^x > x + 1$. As we mentioned in lecture, and as illustrated on page 282 of the textbook, the graph of $y = e^x$ has a slope of exactly 1 at the point $(0, 1)$. The line $y = x + 1$ also passes through the point $(0, 1)$ with slope 1, and that is the only place where the two graphs touch. Everywhere except at $x = 0$, the graph of $y = e^x$ is above the graph of $y = x + 1$, and so the domain is all real numbers $x \neq 0$.

$$\text{Domain: } (-\infty, 0) \cup (0, \infty)$$