Math 16A, Fall 2000, Professor Harrison
Solutions to Practice Quiz, 7 December 2000

1. (a) Is the following statement True or **False**? (circle one):
If $f'(x) = 3x^2$, then $F(2) = 8$.

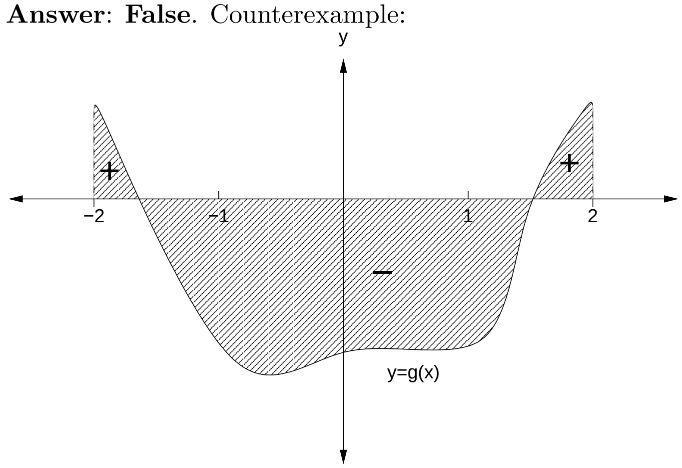
(b) (4 points) If you said it was true, explain why. If you said it was false, come up with a specific counterexample which disproves it.

The statement $f'(x) = 3x^2$ says that $f(x)$ is an antiderivative of $3x^2$. (Notice that I didn't say "*the* antiderivative".) The first function that comes to mind is x^3, but remember that **antiderivatives are only unique up to a constant**.

Answer: **False**. Counterexample: $f(x) = x^3 - 1$
$f'(x) = 3x^2$, but
$f(2) = 2^3 - 1 = 8 - 1 = 7$, not 8.

2. (a) Is the following statement True or **False**? (circle one):
If $g : [-2, 2] \to \mathbb{R}$ is a smooth function such that $g(-2) > 0$ and $g(2) > 0$, then $\int_{-2}^{2} g(x)\,dx > 0$.

(b) (4 points) If you said it was true, explain why. If you said it was false, come up with a specific counterexample (formula or graph) which disproves it.
Answer: **False**. Counterexample:



As you can see, $g(-2)$ and $g(2)$ are both positive, but since the shaded area below the x-axis is larger than the area above the x-axis, the total value of the integral $\int_{-2}^{2} g(x)\,dx$ is negative.

For a different counterexample given by a formula, try $g(x) = x^2 - 3$. We can check that it makes the hypotheses true and the conclusion false by doing some calculations: $g(-2) = (-2)^2 - 3 = 4 - 3 = 1$ is positive, and so is $g(2)$, but

$$\begin{aligned}
\int_{-2}^{2} (x^2 - 3)\,dx &= \left(\frac{x^3}{3} - 3x\right|_{-2}^{2} \\
&= \left(\frac{8}{3} - 6\right) - \left(\frac{-8}{3} - (-6)\right) \\
&= \left(\frac{8}{3} - \frac{18}{3}\right) - \left(\frac{-8}{3} + \frac{18}{3}\right) \\
&= \frac{8 - 18 + 8 - 18}{3} = -\frac{20}{3},
\end{aligned}$$

which is negative.

3. (8 points) One of the following functions is an antiderivative of $\ln(x)$ for $x > 0$. Which is it? Justify your answer.

a) $\frac{1}{2}\left(\ln(x)\right)^2$
b) $\ln\left(x^2/2\right)$
c) $2 - x + x\ln(x)$
d) $1/x$

Answer: **c)**

This is just a matter of carefully taking the derivative of each of these functions and seeing which one gives us $\ln(x)$.

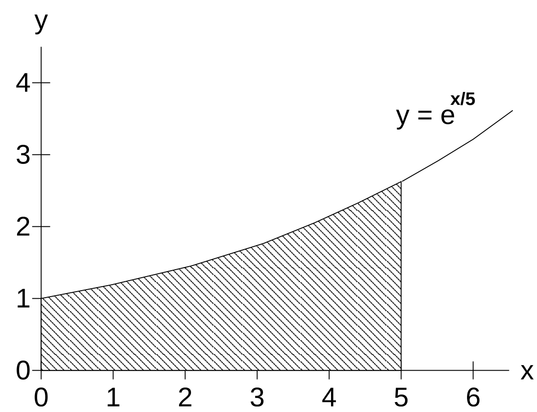
a) $\frac{d}{dx}\left[\frac{1}{2}\left(\ln(x)\right)^2\right] = \frac{2}{2}(\ln(x)) \cdot \frac{d}{dx}\left[\ln(x)\right] = \ln(x)/x$. Don't forget to multiply by the derivative of the inside function when using the chain rule.

b) $\frac{d}{dx}\left[\ln\left(x^2/2\right)\right] = 1/(x^2/2) \cdot \frac{d}{dx}\left[x^2/2\right] = 2/x^2 \cdot x = 2/x$. How can this function be an antiderivative of $2/x$ when $2\ln(x)$ is also an antiderivative of $2/x$? See if you can figure it out using the properties of logarithms...

c) $\frac{d}{dx}\left[2 - x + x\ln(x)\right] = 0 - 1 + \left(\frac{d}{dx}x\right)\ln(x) + x\left(\frac{d}{dx}\ln(x)\right) = -1 + \ln(x) + x/x = -1 + \ln(x) + 1 = \ln(x)$. This is the correct answer.

d) $\frac{d}{dx}\left[x^{-1}\right] = -x^{-2} = -1/x^2$. This function is the *derivative* of $\ln(x)$, not an *antiderivative* of it.

4. (8 points) Use the Fundamental Theorem of Calculus to find the area of the shaded region below the graph of $y = e^{x/5}$.



To answer this problem, we will need to know an antiderivative for $e^{x/5}$. This would be easy if e^x were the function in the graph, since e^x is its own derivative (and antiderivative). Maybe $e^{x/5}$ acts the same way, or close to the same way—let's take $e^{x/5}$ itself as our first guess for the antiderivative of $e^{x/5}$. We can check how close we are by taking the derivative of our guess:

$$\frac{d}{dx}\left[e^{x/5}\right] = e^{x/5} \cdot \frac{d}{dx} x/5 = \frac{1}{5} e^{x/5}$$

This is too small by a factor of 5, so we just multiply our first guess by 5:

$$\frac{d}{dx}\left[5e^{x/5}\right] = 5e^{x/5} \cdot \frac{d}{dx} x/5 = \frac{5}{5} e^{x/5} = e^{x/5}$$

Thus $5e^{x/5}$ is the function we need to solve the problem.

Answer: Since the shaded region is entirely above the x-axis, its area is given by the following definite integral:

$$\int_0^5 e^{x/5}\,dx = \left(5e^{x/5}\right|_0^5 = (5e^{5/5}) - (5e^{0/5}) = 5e - 5.$$

It is also possible to find the area of the region in another way (which on a deeper level is actually equivalent to what we did). Think about the following questions:

Given any function $f(x)$, how are the graphs of $y = f(x)$ and $y = f(x/5)$ related?

How does this change affect the areas of regions?

Do you see why the answer we got is exactly 5 times $\int_0^1 e^x\,dx$?