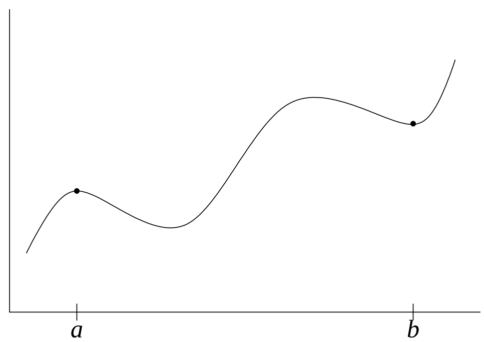
Math 16A, Fall 2000, Professor Harrison
Solutions to page 2 of Midterm II, 29 November

4. (1 point)
Version 1: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. If f has a relative maximum at $x = a$ and a relative minimum at $x = b$ then $f(b) \leq f(a)$.

Version 2: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. If f has a relative minimum at $x = a$ and a relative maximum at $x = b$ then $f(a) \leq f(b)$.

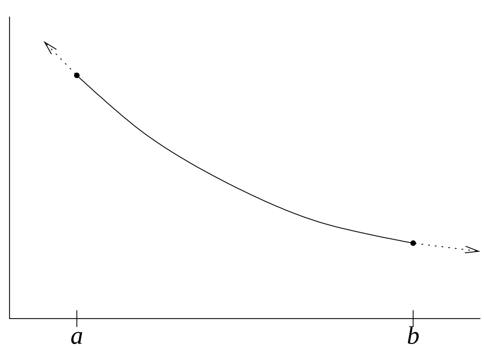
Answer: **False**. Possible counterexample:



(For a counterexample to Version 2, switch the labels a and b.)

Some students misinterpreted the symbol "$\leq$", apparently reading the conclusion as "it is possible for $f(b)$ to be less than $f(a)$ and it is also possible for $f(b)$ to be equal to $f(a)$" (which would be true, by the way). That's not what it says at all; the statement $3 \leq 5$ is definitely *true* even though 3 and 5 can never be equal. The only way invalidate the conclusion $f(b) \leq f(a)$ is to find an example where $f(b)$ is actually *greater* than $f(a)$, such as above.

5. (1 point) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. If f has two relative extrema in the interval $[a, b]$ then f has an inflection point in $[a, b]$.
Answer: **False**. Possible counterexample:

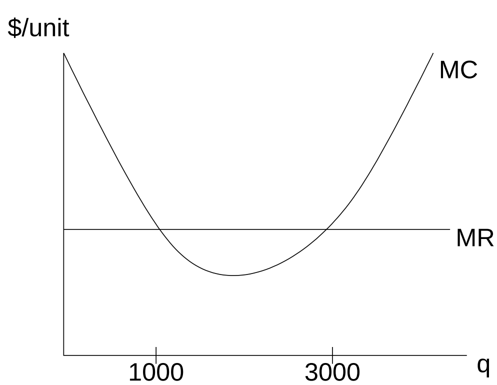


Note: The definition of *relative extremum* given in your textbook is different from the one used by most mathematicians, including Professor Harrison. The standard definition of a relative maximum, which will apply for the rest of this course, *includes* local maxima which may occur at the endpoints of an interval. In the figure above, the left endpoint is a relative maximum because no number in the interval which is near to a gives a higher value when plugged into the function. This is the definition which you will need to know for the final. For example, do you understand why every point of a constant function is both a relative maximum and a relative minimum?

Unfortunately, all four of us (the GSIs) read the nonstandard definition in the textbook and repeated it in section. Although your primary source of information should be the lectures, not the textbook or section, for this one problem we decided to give credit for a carefully reasoned response which used the wrong definition but was otherwise consistent and logically valid. (As always, no credit was given for merely repeating the statement.)

Answer using wrong definition: **True**. A relative extremum is where the slope of the graph changes signs. [$\longleftarrow$This is the wrong definition.] If it first changes from positive to negative, then the next change must be from negative back to positive, and vice versa, so there must be both a relative minimum and a relative maximum in the interval. At a relative minimum the graph is concave up, since the graph is above the tangent line, and similarly at a relative maximum the graph is concave down. Thus at some point the graph must change from concave up to concave down (or vice versa), giving us a point of inflection.

6. (2 points) The figure below shows graphs of marginal cost and marginal revenue. Estimate the production levels that could maximize profit. Explain your reasoning.



This is an optimization problem, so we need to first find the points where the derivative is zero or undefined (in other words, the critical points) and then decide which of these points and which of the endpoints is a relative maximum (rather than a relative minimum or neither).

Answer: Profit is revenue minus cost, or $\pi(q) = R(q) - C(q)$, and so $\frac{d}{dq}[\pi(q)] = MR - MC$. The critical points of total profit thus occur when $\pi'(q) = 0$ and $MR = MC$. This happens at two points: one near $q = 1000$ and one near $q = 3000$. Near $q = 1000$, $MR - MC$ goes from negative to positive, so this point could not possibly be a maximum (it's a local minimum). [Another way to see this is to look at $\pi''(q) = MR' - MC'$, which is positive at $q = 1000$.] However, near $q = 3000$, the slope of total profit goes from positive to negative, representing a local maximum, so this could be the point which maximizes profit.

Now for the endpoints. To the right of the graph marginal cost is greater than marginal revenue, so continuing to increase production would result in greater and greater losses: we should reduce production down to $q = 3000$, and there can be no maximum after that point. The same is true near the left endpoint: increased production results in greater losses, so near $q = 0$ we should reduce production. That means that the endpoint $q = 0$ represents another relative maximum of profit. There are two production levels that could maximize profit, one at $q = 0$ and another near $q = 3000$.

Some students failed to recognize the difference between cost and *marginal* cost, etc., and tried to find the point on the graph where MC reaches the lowest point below MR. That's maximizing *marginal* profit, not profit. What we want is $MR = MC$, which similarly is *not* where the two curves shown have equal slope. Although marginal cost would be represented by slope on a graph of $C(q)$, what we have here is a graph of $MC(q)$ itself, so $C'(q) = MC(q)$ is just the height of the curve MC above the q-axis, and $MR = MC$ where the two curves cross. If we were to look at the *slope* of a graph of *marginal* cost, that's two derivatives—$C''(q)$—which is useful for determining concavity, but not for the first step of finding the critical points of the profit function.

The problem does not ask to determine which of these two points represents the actual maximum profit, since that requires using the Fundamental Theorem of Calculus, which was not part of what was tested on this midterm. Just to satisfy our curiousity, though, let's see how we can answer that question, too.

We start as always with the fact that total profit is total revenue minus total cost. If this were a graph of revenue and cost, the profit would thus be represented by a vertical distance on the graph. Instead, this is graph of the *derivatives* of revenue and cost. The FTC tells us that the *value* (height above the q-axis) of an antiderivative (in this case, total profit) is related to an *area* in the graph of the thing it's an antiderivative of (in this case, marginal profit). So profit is the area *below* the MR line and *above* the MC curve.

In the first part of the graph, things are switched from what we would like: The triangular region is below MC and above MR, so it represents a negative profit, also called a loss. The lens-shaped area between $q = 1000$ and $q = 3000$ is above MC and below MR, so the marginal profit is positive in this region and the area represents a profit. This may or may not be enough to offset the losses from $q = 0$ to $q = 1000$. In fact, a careful inspection reveals that the lens-shaped region can be placed entirely inside of the triangular region, so the profit is not enough to compensate for the earlier losses and the overall profit is negative at our local maximum, $q = 3000$. We're better off not making any product at all. It's simply not a profitable enterprise, especially considering whatever startup costs there may be (which we can't guess from the graph). Now, aren't you glad you learned about the Fundamental Theorem of Calculus before someone tried to sell you stock in this company?