

Math 16A, Fall 2000, Professor Harrison
Solutions to Quiz #6 (mock quiz), 27 October

1. (a) (0 points) Is the following statement True or **False**? (circle one):

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function and $f(x) > 0$ for all real numbers x , then $f''(x) \geq 0$ for all real numbers x .

(b) (8 mock points) If you said it was true, explain why. (Don't just repeat the statement and then say it must be true!) If you said it was false, come up with a specific counterexample (formula or graph) which disproves it.

Answer: False. Draw any smooth curve which sometimes bends down towards the x -axis, but which never touches or crosses it. The function $f(x) = \cos(x) + 2$ is a good example. We haven't actually learned how to differentiate trigonometric functions yet, but the graph itself is enough for the counterexample: f is smooth and always positive, but the graph of $f(x)$ is bending down at $x = 0$, so $f''(0)$ is negative.

If you'd like an example with a formula you know how to differentiate, start with a function $g(x)$ which is always positive but which has a local minimum at 0, with $g''(0) > 0$, and then let $f(x) = 1/g(x)$. The local minimum turns into a local maximum (why?), and $f''(0) < 0$. It's a useful exercise to do this calculation with an actual formula, or better yet to prove for any function starting with $g(0) > 0$, $g'(0) = 0$, and $g''(0) > 0$, and then differentiating $1/g(x)$ twice.

2. (8 mock points) There is an important function in statistics which is only defined in terms of its derivative: $h'(x) = ae^{-x^2}$, where a is a positive constant, and $h(0) = 0$. The actual function $h(x)$ cannot be written down using anything we have learned, but we can still answer many questions about $h(x)$ just by knowing its derivative and the value of the function at zero.

(a) When is $h(x)$ increasing?

Answer: The number e raised to any power, positive or negative, gives a positive result. (Remember that e^{-x^2} is just $1/e^{x^2}$.) Since a is also a positive number, $h'(x)$ is always positive, so $h(x)$ is increasing for all x .

(b) When is $h(x)$ concave upwards?

Answer: To find the second derivative of $h(x)$, we differentiate $h'(x)$. The chain rule gives us

$$\frac{d}{dx} [ae^{-x^2}] = ae^{-x^2} \frac{d}{dx} [-x^2] = -2axe^{-x^2},$$

which is positive exactly when x is negative. So the graph of $h(x)$ is concave upwards on the interval $(-\infty, 0)$ and nowhere else. (If only it were so easy to get the axe from Stanford!)

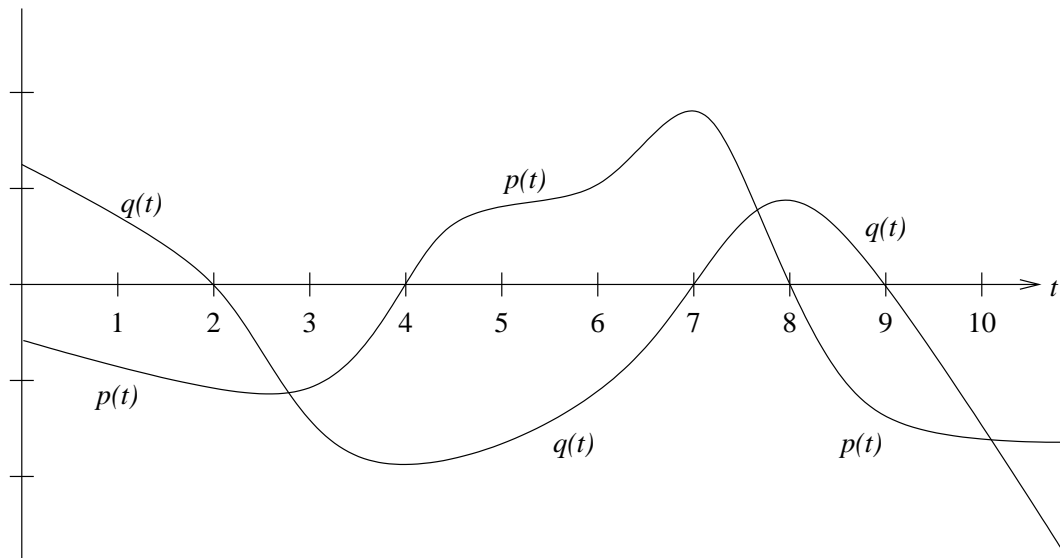


FIGURE 1. $m'(t)$ and $m''(t)$

(c) Is $h(-1)$ positive or negative?

Answer: Since $h(x)$ is increasing everywhere, and in particular on the interval from -1 to 0 , and since $h(0) = 0$, the function must increase to 0 from a *negative* value at $x = -1$. Hence, $h(-1) < 0$.

(d) At what points does $h(x)$ have local maxima or minima?

Answer: We know that for a differentiable function, the derivative is always zero at a local maximum or minimum (although the converse isn't necessarily true). Since $h'(x)$ is never zero, there can't be any local maxima or minima at all.

3. (8 mock points) The mercury concentration in a certain lake is given by a function $m(t)$, where t is the number of years since January 1990. Figure 1 above does NOT show $m(t)$; instead, it shows the first and second derivatives, $m'(t)$ and $m''(t)$.

(a) Which one is $m''(t)$: the function labelled $p(t)$ or the function labelled $q(t)$?

Answer: Since $m''(t)$ is the derivative of $m'(t)$, we just need to figure out which of these graphs has its value equal to the slope of the other graph. Checking a few different t -values we see that the value of $p(t)$ is equal to the slope of $q(t)$, rather than the other way around, so $m'(t) = q(t)$ and $m''(t) = p(t)$.

(b) Did the concentration of mercury increase or decrease from January 1998 to January 1999?

Answer: The slope of $m(t)$, which is equal to the value of $q(t)$, is positive between $t = 8$ and $t = 9$, so the concentration of mercury was increasing during that year.

(c) At $t = 7$, is the graph of $m(t)$ concave up, concave down, or neither?

Answer: The concavity of $m(t)$ at $t = 7$ is given by the value of $p(t)$. Since $p(7)$ is above the t -axis, the graph of $m(t)$ must be concave up at $t = 7$. We could also have answered this problem by looking at the slope of $q(t)$.

(d) Name any times at which there was a local minimum in the mercury concentration.

Answer: A local minimum of $m(t)$ happens when the slope goes from negative to positive, or in other words when the value of $q(t)$ crosses from below the t -axis to above it. This happened at $t = 7$, in January of 1997.

(e) (extra mock credit) At what time between January 1990 and the present was the mercury concentration the very worst (highest)?

Answer: To answer this question we need to check all of the relative extrema, which occur at $t = 2$, $t = 7$, and $t = 9$, and also check the endpoints. We have already seen that the value of $m(t)$ at $t = 7$ is a relative minimum (rather than a relative maximum), so we can cross it off of the list of things to check. How about the endpoints? In 1990 the concentration was increasing, so 1990 can't be the absolute maximum. At the current time the concentration is decreasing, so now can't be the absolute maximum, either.

We are left with two choices, the relative maxima at $t = 2$ and at $t = 9$. None of the tools we have used so far give us any way of choosing between these two, so we really have to see the graph of $m(t)$ to know when the concentration was higher. Fortunately, we have all the information we need to reconstruct what $m(t)$ looks like, given any starting point. All we are asking is which value is greater, $m(2)$ or $m(9)$, and any starting point we choose will give us the same answer to that question, so it doesn't matter that we don't know the real starting point. By carefully drawing a graph whose slope is always equal to the value of $q(t)$, you should be able to convince yourself that $m(2)$ is greater than $m(9)$: the worst concentration of mercury during the past ten years occurred in January of 1992.

Later in the course we will learn a powerful technique which allows us to immediately answer questions like part (e) just by measuring *areas* in the graph.