Math 16A, Fall 2000, Professor Harrison
Partial Solutions to True / False Practice Sheet (for Midterm I)

Many of these answers are incomplete, and would not be given full credit as written. You should try to flesh out the details of proofs on your own, and you should similarly try to come up with specific, concrete counterexamples following the suggestions given.

Please note that some of these examples were deliberately made difficult, to get you thinking. The true / false questions on the midterm will probably not be quite as tricky as the worst of these, so don't second-guess yourself on the midterm if your first answer seems too easy. On the other hand, do make sure 1) that you thoroughly understand exactly what the question is asking, 2) that you know the exact definitions of all the words and notation, and 3) that you aren't assuming anything that isn't explicitly stated.

1. False; $\ln(x+y)$ cannot be simplified any further without knowing what x and y are. Idea for counterexample: Plug in just about numbers for x and y. Show that the results aren't equal by exponentiating.

2. True. Proof: $\ln(xy) = \ln(x) + \ln(y)$ because

$$e^{\ln(xy)} = xy = e^{\ln x} e^{\ln y} = e^{\ln(x)+\ln(y)}.$$

3. False; $\ln(x/y)$ simplifies to $\ln(x) - \ln(y)$ instead. Counterexample: Just about any numbers will do. Try some.

4. True. There are several ways to derive this from the rules we already know for differentiation; one way is to use the product rule twice, remembering that the derivative of a constant is zero.

5. True. Idea of proof: Plug in the definitions of $f'(a)$ and "continuous at a", and try to see how to get from one to the other. Hint: Start with the limit definition of $f'(a) = b$ and rewrite it by letting $x = a + h$. Multiply both sides by the denominator and try to end up with the definition of continuity.

6. False. In the hypothesis, b is an output of f rather than an input, so who knows what happens near $f(b)$? Idea for counterexample: Use a discontinuous function, like the Heaviside function.

7. True. You can think of x as $1/n$ and see that this reduces to a limit we already know. However, technically this isn't enough since it only tells us about x-values which are the reciprocals of counting numbers, not all x-values near 0. Idea of proof: Show that

$$\lim_{h\to 0} \ln\left[(1+h)^{\frac{1}{h}}\right] = 1$$

using the limit definition of the derivative of $\ln(x)$ and the fact that you know what that derivative is.

8. False. The true statement is about the derivative of $f(x)+c$, which is not the same as $f'(x+c)$. Counterexample: Try just about any non-linear function and non-zero constant.

9. False. Just because most of the examples we've seen of sequences which converge happen to be increasing or decreasing, don't assume that they all are. In fact, in some sense increasing sequences and decreasing sequences are "rare". Simple counterexample: Let x_n be a constant sequence. Slightly more interesting counterexample: Let $x_n = (-\frac{1}{2})^n$.

You may want to try your hand at finding a counterexample to the following statement, which is also false:

If x_n is a sequence which converges to the real number L, but $x_n < L$ for all n, then x_n is an increasing sequence.

10. True. Some people were confused by "id". That's just the symbol for the identity function, which takes an input and doesn't change it at all: $\text{id}(x) = x$ by definition. Proof: For all x in the domain of g, $g(x) = h(f(g(x))) = h(x)$. Similarly for all x in the domain of h.

11. False. This is a little tricky; it would be true if it said that f is continuous (even just at a), or that $f(a) = b$, or even that x_n is never equal to a. Idea for counterexample: Take a continuous function and make it discontinuous by redefining it at a single point a. Then let x_n be the constant function $\{a, a, a, \ldots\}$.

12. False, for the same technical reasons we complained about in problem 7. It would be true if it also said that $f(x)$ is continuous. One counterexample: Let $f(x)$ be our favorite discontinuous function, the Heaviside function, and let x_n be a decreasing sequence which converges to 0.

13. False. It would be true if it were stated the other way around. Idea for counterexample: Let $f(x)$ be a constant function and let $g(x)$ be any discontinuous function. It doesn't matter what $g(x)$ is, $f(g(x))$ will still be a constant function and hence continuous.

14. False. The best you can say about $g \circ f$ is that it will be continuous. Idea for counterexample: Let f be the identity function $f(x) = x$ and let g be any function which is continuous but not differentiable.

15. False. It looks like $f(x)$ is just the inverse of e^x, or in other words $\ln(x)$, but it only satisfies half of the definition of inverse: we would need $e^{f(x)} = x$ as well. In particular, the equation $f(e^x) = x$ only tells us what f does to *positive* numbers, since e^x is always positive. Idea for counterexample: Define $f(x)$ in pieces, so that it equals $\ln(x)$ when $x > 0$, but does something completely different when x is negative. Verify the hypothesis, and then see what happens to the derivative when x is negative.