Math 128a, Section 3 — Midterm Problems

(1) The mean M and variance V of n numbers x_i are defined by

$$M = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad V = \frac{1}{n}\sum_{i=1}^{n}(x_i - M)^2 = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - M^2$$

where the two formulas for V are equivalent in exact arithmetic because

$$\sum_{i=1}^{n}(x_i - M) = 0.$$

Take $n = 2$ for simplicity, and show that in floating-point arithmetic with machine precision ϵ the two formulas satisfy different error bounds

$$|V - \mathrm{fl}\left(\frac{1}{2}\sum_{i=1}^{2}(x_i - M)^2\right)| \le 2\epsilon|V| + O(\epsilon^2)$$

and

$$|V - \mathrm{fl}\left(\left(\frac{1}{2}\sum_{i=1}^{2} x_i^2\right) - M^2\right)| \le 3\epsilon(x_1^2 + x_2^2 + M^2) + O(\epsilon^2)$$

Which formula has a better guaranteed accuracy in floating-point arithmetic?

(2) Suppose that a numerical integration rule

$$\int_0^1 f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

with points $x_i \in [0,1]$ is exact for polynomials of degree d. Suppose that f can be approximated by a polynomial p of degree d to accuracy ϵ on the interval $[0,1]$:

$$\max_{0\le x\le 1} |f(x) - p(x)| \le \epsilon.$$

Show that the error in the numerical integration rule applied to integrate f is bounded by

$$|\int_0^1 f(x)dx - \sum_{i=1}^{n} w_i f(x_i)| \le \Omega\epsilon$$

where

$$\Omega = 1 + \sum_{i=1}^{n} |w_i|.$$

(3) Compute the Newton form of the cubic polynomial which interpolates the function $f(x) = |x - 1|$ at the points $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 0$.

(4) Suppose we define orthogonal polynomials with respect to a different inner product

$$f^* g = \int_{-1}^{1} f(x) g(x) \frac{dx}{\sqrt{1 - x^2}}.$$

(a) Compute the first three orthogonal polynomials $p_0(x) = 1$, $p_1(x) = x$, and $p_2(x)$. (Hint: put $x = \cos(t)$ to evaluate the integrals.)
(b) Find the points of the 2-point Gaussian quadrature formula for evaluating weighted integrals of the form

$$\int_{-1}^{1} f(x) \frac{dx}{\sqrt{1 - x^2}}.$$

(5) **(a)** Consider the `matlab` code

```
x=1; q=0;
while x>0
   x=x/2;
   q=q+1;
end;
```

In binary floating-point arithmetic, what does it evaluate?
(b) Consider the `matlab` code

```
x=1;
while 1 + x > 1
   x=x/2;
end;
```

In binary floating-point arithmetic, what does it evaluate?
(c) Consider the `matlab` code

```
function a = f(x,y,z)
n = length(x);
for k = 1:n-1
   y(k+1:n) = (y(k+1:n)-y(k:n-1)) ./ (x(k+1:n) - x(1:n-k));
end
a = y(n)*ones(size(z));
```

```
for k=n-1:-1:1
   a = (z-x(k)).*a + y(k);
end
```

In exact arithmetic, what would it evaluate?

(6) **(a)** Find the Lagrange form of the quadratic polynomial interpolating the function $f(x) = x^3$ at the points $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.
(b) Find a numerical integration rule of the form

$$\int_0^3 f(x)dx = af(1) + bf(2) + cf(3)$$

which is exact whenever f is a polynomial of degree 2. Note that the lower limit of integration is 0, not 1.
(c) Give weights w_1, w_2 and w_3 such that the error $E(h)$ in the approximation

$$\int_0^{3h} f(x)dx = \sum_{i=1}^{3} w_i f(ih) + E(h)$$

is of order $E(h) = O(h^2)$.

(7) **(a)** Find the power form of the linear polynomial $p(x)$ interpolating the function $f(x) = x^2$ at the points $x_1 = 1$ and $x_2 = 2$.
(b) Show that the absolute error in evaluating $p(x)$ at an arbitrary point x by floating-point arithmetic with machine precision ϵ is bounded by

$$|p(x) - \mathrm{fl}(p(x))| \leq (6|x| + 2)\epsilon.$$

(c) Show that $\mathrm{fl}(p(x)) = p(\hat{x})$ is equal to the exact value of the polynomial p at a point $\hat{x}$ such that $|x - \hat{x}| \leq (1 + 2|x|)\epsilon$.

(8) Let k be an arbitrary positive integer and consider interpolating $\sin(kx)$ by a polynomial of degree 10 at 11 equispaced points on the interval $[0, 1]$. Give a bound for the error $|\sin(kx) - p(x)|$ as a function of k when x lies in the interval $[0.0, 0.1]$.

(9) Let

$$\varphi_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

be the ith Lagrange basis function for interpolation at n points $x_1, \ldots, x_n$. Show that

$$\alpha_k(x) = (1 - 2\varphi_k'(x_k)(x - x_k))\varphi_k(x)^2$$

and

$$\beta_k(x) = (x - x_k)\varphi_k(x)^2$$

are basis functions for Hermite interpolation of a function f and its derivative f' at the points $x_1, \ldots, x_n$.

(10) Suppose a function f can be evaluated with relative error bounded by $\delta < 1/2$. We approximate its derivative by the forward difference quotient

$$D_h f(x) = \frac{f(+h) - f(x)}{h}.$$

(a) Use Taylor expansion to show that in exact arithmetic

$$|D_h f(x) - f'(x)| \leq |\frac{M_2}{2} h^2$$

whenever $|f''(x)| \leq M_2$ for all x.
(b) Show that in floating-point arithmetic with machine epsilon $\epsilon < 1/2$ the error in evaluating the approximation is bounded by $6M_0(\epsilon + \delta)/h$ whenever $|f(x)| \leq M_0$ for all x.
(c) Find h (as a function of $\epsilon + \delta$, M_0 and M_2) which gives the best error bound on the approximation of $f'(x)$ by $\mathrm{fl}(D_h f(x))$.

(11) Consider two sets of data

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad y_1 = 1, \quad y_2 = 12, \quad y_3 = 13,$$

and

$$\hat{x_1} = 3, \quad \hat{x_2} = 2, \quad \hat{x_3} = 1, \quad \hat{y_1} = 13, \quad \hat{y_2} = 12, \quad \hat{y_3} = 1.$$

Prove or disprove: the same quadratic polynomial $p(x)$ interpolates both sets of data.

(12) Describe three techniques for proving the following theorem: given n distinct points $x_i \in R$ and n values $y_i \in R$, there is a unique polynomial p of degree $n - 1$ such that $p(x_i) = y_i$ for $i = 1 : n$.

(13) Suppose $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Describe two methods for evaluating p at a given point x. Show that one of the methods is in general both faster and more accurate in floating-point arithmetic.

(14) Verify that $p(x) = 3+2(x-1)+4(x-1)(x+2)$ and $q(x) = 4x^2+6x-7$ both interpolate $q(x)$ at $x = 1$, -2 and 0. Explain why this does not contradict the uniqueness of polynomial interpolation.

(15) The polynomial $p(x) = x^4 - x^3 + x^2 - x + 1$ has values 31, 5, 1, 1, 11, 61 at the points -2, -1, 0, 1, 2, 3. Find a degree-5 polynomial $q(x)$ with values 31, 5, 1, 1, 11, and 30 at the same points.

(16) Let $p(x)$ have degree n. What are the divided differences $p[x_1, x_2, \ldots, x_{n+2}]$ for $n+2$ distinct points x_i? Prove it.

(17) Let $p(x)$ be the polynomial of degree $n-1$ which interpolates $f(x)$ at n points x_i. Show that the derivatives satisfy

$$f'(x_i) - p'(x_i) = \frac{1}{n!} f^{(n)}(\xi) \prod_{j \neq i} (x_i - x_j)$$

at each interpolation points x_i.

(18) Let $|\delta_i| \leq \delta < 1/2$ for $i = 1 : n$.
(a) Show that

$$\prod_{j=1}^{n} (1 + \delta_i) = 1 + \Delta$$

where $|\Delta| \leq 2n\delta$.
(b) Show that

$$\prod_{j=1}^{n} (1 + \delta_i)^{-1} = 1 + \Delta$$

where $|\Delta| \leq 2n\delta$.

(19) Prove that the $n \times n$ matrix with 4's down the diagonal, -1's adjacent to the diagonal and 0's elsewhere is invertible for any n.

(20) Suppose x and y are floating-point numbers and the floating-point subtraction $\mathrm{fl}(x - y)$ evaluates to 0. Prove or give a counterexample: $x = y$.

(21) Define the machine precision ϵ for floating-point arithmetic. Prove that the model of floating-point arithmetic which requires that the floating-point result of the addition $x + y$ be given by the exact result correctly rounded satisfies the relative error bound

$$\frac{|x + y - \mathrm{fl}(x + y)|}{|x + y|} \leq \epsilon$$

as long as no overflow or underflow occurs and $x + y \neq 0$.

(22) Use the method of undetermined coefficients to derive a numerical integration formula

$$\int_0^1 f(x)dx = w_0 f(0) + w_1 f(1) + w_2 f(2)$$

which is exact for polynomials of as high degree d as possible, and determine the maximal degree d. Without any additional work determine a comparable rule of the form

$$\int_0^1 f(x)dx = u_0 f(0) + u_1 f(1) + u_2 f(-2).$$

(23) Show that for any function f on the interval $[-1, 1]$ and any nonzero s, there exists a unique function of the "exponential spline" form

$$p(t) = a + bt + c\cosh(st) + d\sinh(st)$$

such that

$$p(-1) = f(-1), \quad p(+1) = f(+1), \quad p'(-1) = f'(-1), \quad p'(+1) = f'(+1).$$

(24) Show that for any function f on the interval $[-1, 1]$ and any $s \in (0, \pi]$, there exists a unique function of the "trigonometric spline" form

$$p(t) = a + bt + c\cos(st) + d\sin(st)$$

such that

$$p(-1) = f(-1), \quad p(+1) = f(+1), \quad p'(-1) = f'(-1), \quad p'(+1) = f'(+1).$$