ISOMORPHISMS OF DIFFERENTIAL FORMS AND COCHAINS

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Abstract. This paper proves an isomorphism theorem for cochains and differential forms, before passing to cohomology. De Rham's theorem is a consequence. This leads to an extension of much of calculus and homology theory to nonsmooth domains, called *chainlets* and makes available combinatorial techniques for smooth domains that limit to the classic analytic methods. We find maximal subspaces of L^1 forms that satisfy Stokes' theorem for domains of chainlets giving a measurable, as well as optimal, extension of the theory.

1. Introduction

A differential form acts as a linear functional on the vector space of simplicial chains via integration and is therefore a cochain. De Rham's theorem tells us the integration mapping Ψ sending L^1 differential forms into cochains

$$\Psi(\omega) \cdot A = \int_A \omega$$

induces an isomorphism of cohomology rings. This paper is a study of the mapping Ψ. Theorem A shows that Ψ is an isomorphism at the level of smooth differential forms and cochains before passing to cohomology. Since Ψ commutes with the exterior derivative on forms and the codifferential operator on cochains, de Rham's theorem is a direct corollary. Poincaré duality, which relies on de Rham's theorem, extends to chains and forms [H5].

In the 1950's Whitney [W] and Wolfe [Wo] identified spaces of cochains isomorphic to Lipschitz and flat differential forms, respectively. These isomorphisms allow one to move between the infinitesimal and the global, providing a link between analysis and geometry. The link is tenuous, however, since basic operators on differential forms such as the exterior derivative, Hodge star, and pullback are not all continuous nor even defined in these spaces.

The duality between "smoothness" of mappings and "roughness" of invariant sets appearing in the C^2 Seifert conjecture counterexamples * of [H1] suggested the isomorphisms presented here. Our isomorphisms *do* preserve the basic operators, enriching the links between topology, measure theory,

*This duality is not present in the C^∞ Seifert counterexamples of Kuperberg [K].

dynamical systems, fractal geometry, and physics. A common language between these subjects emerges with common operators and cross fertilization of results.

We make use of definitions, notation, and results from [H3] throughout, assuming $0 < \alpha \le 1$ and r is an integer. Central to the construction of the isomorphisms is a 1-parameter family of norms $|A|_{r,\alpha}$ on p-dimensional polyhedral chains P supported in $\mathbb{R}^n$, introduced in [H3]. The Banach spaces obtained on completion are denoted $\mathcal{A}^{r,\alpha} = \mathcal{A}_p^{r,\alpha}$ with elements called *chainlets*. The dual spaces of co-chainlets X are denoted $(\mathcal{A}^{r,\alpha})^*$ with dual norms $|X|_{r,\alpha}$. We identify the integer r with $(r-1,1)$ and order pairs (r,α) lexicographically. Let $\mathcal{B}^{r,\alpha}$ denote the space of differential forms of class $C^{r,\alpha}$ with bounded norm. Theorem A shows the spaces $(\mathcal{A}_p^{r,\alpha})^*$ and $\mathcal{B}^{r,\alpha}$ are isomorphic. We assume throughout that $(r,\alpha) > 0$, unless otherwise stated.

Expositors of advanced calculus often describe smooth differential forms through their properties as cochains. For example, this approach is used extensively in [E] as an intuitive guide, although Edwards wrote that one could not actually define differential forms as cochains. This conceptually simple approach was hampered by a lack of a definition of differentiability classes of cochains. Theorem A implies

> A differential form can be defined as a bounded linear functional X on polyhedral chains. Its exterior derivative may be defined through the action of X on boundaries of polyhedral chains. It is (r,α)-*smooth* if there exists $C > 0$ such that
>
> $$X \cdot P \le C|P|_{r,\alpha}$$
>
> for all polyhedral P.

Smooth cochains extend to co-chainlets. These definitions of smooth forms and their exterior derivatives require no pointwise information such as partial derivatives. Theorem A shows these definitions are equivalent to the analytical definitions as multi-linear functionals on ordered p-tuples of vectors based at points with coordinate functions of class $C^{r,\alpha}$.

Measurable integrands

We recall the subspaces $\mathcal{J}^{r,\alpha}$ of L^1 differential forms ω with norm $|\omega|_{r,\alpha}$ defined in [H3]. Each form $\omega \in \mathcal{J}^{r,\alpha}$ is a.e. equal to a smooth form in $\mathcal{B}^{r,\alpha}$, but this is not a sufficient condition to be in $\mathcal{J}^{r,\alpha}$. Our second main result, Theorem B, proves the following diagram commutes

$$\begin{array}{ccccc}
\left(\mathcal{A}_p^{r,\alpha}\right)^* & \xrightarrow{d} & \left(\mathcal{A}_{p+1}^{r-1,\alpha}\right)^* \cdots & \xrightarrow{d} & \left(\mathcal{A}_{p+r}^{\alpha}\right)^* \\
\Psi \quad \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\
\mathcal{J}_p^{r,\alpha} & \xrightarrow{d} & \mathcal{J}_{p+1}^{r-1,\alpha} \cdots & \xrightarrow{d} & \mathcal{J}_{p+r}^{\alpha}
\end{array}$$

and shows the chain complex

$$\mathcal{J}_p^{r,\alpha} \xrightarrow{d} \mathcal{J}_{p+1}^{r-1,\alpha} \cdots \xrightarrow{d} \mathcal{J}_{p+r}^{\alpha}$$

is the maximal chain complex of L^1 forms for which Ψ is an epimorphism of chain complexes.

A consequence is an optimal generalization of Stoke's theorem. (See Theorem 2.8 below.) In [H3], Corollary 4.12, the author proves an extension of Stokes' theorem to chainlet domains and smooth integrands. The extension in this paper to nonsmooth integrands is subtle since one may freely alter smooth forms only on certain p-dimensional subsets of $\mathbb{R}^n$ of Lebesgue measure zero without changing some integrals. (See the examples in §2 .) In a sequel [H4] we prove that all currents with compact support are represented uniquely as chainlets, demonstrating that our extension of Stoke's theorem is optimal. André Weil wrote in his autobiography ([We] pp. 99-100) that it was Henri Cartan's discussions with him about the domain of validity of Stokes' formula that led to the birth of Bourbaki.

> One point that concerned him [Henri Cartan] was the degree to which we should generalize Stokes' formula in our teaching. This formula is written as follows:
>
> $$\int_{b(X)} \omega = \int_X d\omega,$$
>
> where ω is a differential form $d\omega$ its derivative, X its domain of integration, and $b(X)$ the boundary of X. There is nothing difficult about this if for example X is the infinitely differentiable image of an oriented sphere and if ω is a form with infinitely differentiable coefficients. Particular cases of this formula appear in classical treatises, but we were not content to make do with these. In his book on invariant integrals, Elie Cartan, following Poincaré in emphasizing the importance of this formula, proposed to extend its domain of validity. Mathematically speaking the question was of a depth that far exceeded what we were in a position to suspect. Not only did it bring into play the homology theory, along with de Rham's theorems, the importance of which was just becoming apparent; but this question is also what eventually opened the door to the theory of distributions and currents and also to that of sheaves. For the time being, however, the business at hand for Cartan and me was teaching our courses in Strasbourg. One winter day toward the end of 1934, I thought of a brilliant way of putting an end to my friend's persistent questioning. We had several friends who were responsible for teaching the same topics in various universities. "Why don't we get together and settle such

matters once and for all, and you won't plague me with your questions any more?" Little did I know that at that moment Bourbaki was born.

2. Isomorphisms of differential forms and cochains

Given a co-chainlet $X \in (\mathcal{A}^{r,\alpha})^*$ we prove there exists a unique differential form $D_X \in \mathcal{B}^{r,\alpha}$ such that $X \cdot P = \int_P D_X$ for all polyhedral chains P.

Definition The *mass* or p-dimensional Hausdorff measure of a polyhedral p-chain P is denoted $|P|_0$. Given $X \in (\mathcal{A}^\alpha)^*$, $q \in \mathbb{R}^n$, and β a p-direction, define

$$D_X(q;\beta) = \lim_{i\to\infty} \frac{X \cdot \sigma_i}{|\sigma_i|_0} \tag{1}$$

where $\sigma_1, \sigma_2, \dots$ is a sequence of p-simplexes containing q whose p-directions are $\beta = \{\sigma_i\}/|\sigma_i|_0$ and whose diameters $\to 0$ as $i \to \infty$. For any simple p-vector $\beta \neq 0$, set

$$D_X(q;\beta) = |\beta|_0 D_X(q, \beta/|\beta|_0).$$

Set $D_X(q, 0) = 0$. For general p-vectors, extend D_X linearly. The next theorem establishes existence and uniqueness of D_X as a differential form.

Characterization of co-chainlets as smooth differential forms

Lemma **2.1.** *Let $\phi(\beta)$ be a real function of simple p-vectors such that*

1. *$\phi(a\beta) = a\phi(\beta)$ for all real a,*
2. *$\sum_{i=0}^{p+1} \phi(\{\sigma_i\}) = 0$ where $\partial\sigma = \sum \sigma_i$.*

Then there is a unique p-covector ω such that $\omega \cdot \beta = \phi(\beta)$ if β is simple.

This standard result can be found in [W] (V,Theorem 9A).

Theorem 2.2. *To each co-chainlet $X \in (\mathcal{A}^{r,\alpha})^*$, there corresponds a unique differential form $D_X \in \mathcal{B}^{r,\alpha}$ such that*

$$X \cdot P = \int_P D_X$$

for all polyhedral chains P.

This correspondence is an isomorphism with

$$|D_X|_{s,\beta} = |X|_{s,\beta}.$$

for all $-1 < (s,\beta) \le (r,\alpha)$ and

$$\|D_X\|_{C^{r,\alpha}} \le |X|_{r,\alpha} \le (p+1)\|D_X\|_{C^{r,\alpha}}.$$

The proof has elements in common with the proof of Wolfe's isomorphism theorem in the flat category [Wo] and Whitney's proof in the sharp category [W].

Proof. Let $r = 0$ and $X \in (\mathcal{A}^\alpha)^*$.

Fix $q \in \mathbb{R}^n$. Let $\varepsilon > 0$. We show there exists $\delta > 0$ such that if $\sigma, \sigma' \subset B_\delta(q)$ are simplexes with the same p-direction ν then

$$\left| \frac{X \cdot \sigma}{|\sigma|_0} - \frac{X \cdot \sigma'}{|\sigma'|_0} \right| \le \varepsilon. \tag{2}$$

Let $C = |X|_\alpha$ and $\delta^\alpha = \frac{\varepsilon}{6C}$ and $\varepsilon_1 = \frac{\varepsilon}{6} \min\{|\sigma|_0, |\sigma'|_0\}$. We approximate σ in the mass norm with sums of cubes contained inside σ, all translates of one another.

That is, there exists a cube $Q \subset \sigma$ containing q with p-direction ν so that the following holds: There are vectors v_i, $i = 1, \dots, s$ such that $Q_i = T_{v_i} Q$ are in σ and are nonoverlapping and mass $|\sigma - \sum Q_i|_0 \le \frac{\varepsilon_1}{|X|_0}$. Thus $|X \cdot (\sigma - \sum Q_i)| \le \varepsilon_1$. Then

$$|X \cdot (Q_i - Q)| \le |X|_\alpha |Q_i - Q|_\alpha \le C\|Q_i - Q\|_\alpha \le C|Q|_0 \delta^\alpha = |Q|_0 \varepsilon/6.$$

Thus

$$\begin{aligned} |X \cdot (\sigma - s \cdot Q)| &\le \left| X \cdot (\sigma - \sum_{i=1}^s Q_i) \right| + \left| \sum_{i=1}^s X \cdot (Q_i - Q) \right| \\ &\le \varepsilon_1 + sC|Q|_0 \varepsilon/6 \\ &\le |\sigma|_0 \varepsilon/3. \end{aligned}$$

Therefore

$$\begin{aligned} |(X \cdot \sigma)|Q|_0 - (X \cdot Q)|\sigma|_0| &\le |X \cdot \sigma - s(X \cdot Q)||Q|_0 \\ &\quad + |X \cdot Q| \; |s|Q|_0 - |\sigma|_0| \\ &\le |Q|_0 |\sigma|_0 \varepsilon/2. \end{aligned} \tag{3}$$

and

$$\left| \frac{X \cdot \sigma}{|\sigma|_0} - \frac{X \cdot Q}{|Q|_0} \right| \le \varepsilon/2. \tag{4}$$

The same inequality holds for σ'. This establishes (2) and existence and uniqueness of $D_X(q; \nu)$. Observe that (2) implies continuity of D_X as a function of q, with fixed ν.

By the definition of D_X we have

$$\frac{|D_X(q + v; \nu) - D_X(q; \nu)|}{|v|^\alpha} = \lim_{|\sigma_i| \to 0} \frac{|X \cdot (T_v \sigma_i - \sigma_i)|}{|v|^\alpha |\sigma_i|} \le C.$$

Therefore $\|D_X\|_{C^\alpha} \le C$, as a function of q.

We next prove

$$X \cdot \sigma = \int_\sigma D_X$$

for any simplex σ. Let $\varepsilon = 6C\delta^\alpha$ where δ is the diameter of σ.

It follows immediately from the definition of D_X, and equation (2) that

$$\left| D_X(q;\nu) - \frac{X \cdot \sigma}{|\sigma|_0} \right| \le \varepsilon = 6C\delta^\alpha \tag{5}$$

for all $q \in \sigma \subset B_\delta(q)$. We use Riemann integration to show

$$X \cdot \sigma = \int_\sigma D_X(q;\nu)dq \tag{6}$$

where ν is the p-direction of σ. Let $\varepsilon > 0$ and subdivide σ into simplexes $\sigma_1, \dots \sigma_s$ with diameter $< \eta$ where $\eta^\alpha = \frac{\varepsilon}{7C|\sigma|_0}$.

Choose $q_i \in \sigma_i$. Apply equation (5) and the fact that $\|D_X\|_{C^\alpha} < C$ as a function of q to deduce

$$\begin{aligned}\left| X \cdot \sigma_i - \int_{\sigma_i} D_X(q;\nu)dq \right| &\le |X \cdot \sigma_i - D_X(q_i;\nu)|\sigma_i|_0| \\ &\quad + \left| \int_{\sigma_i} (D_X(q_i;\nu) - D_X(q;\nu))dq \right| \\ &\le 6C\eta^\alpha|\sigma_i|_0 + |\sigma_i|_0\|D_X\|_{C^\alpha}\eta^\alpha \\ &\le 7C\eta^\alpha|\sigma_i|_0.\end{aligned}$$

Thus

$$\left| X \cdot \sigma - \int_\sigma D_X(q;\nu)dq \right| \le \varepsilon.$$

Since this holds for all $\varepsilon > 0$, equation (6) follows.

We apply Lemma 2.1 to prove that D_X a differential form. Let σ be a $(p+1)$-dimensional simplex with $\partial\sigma = \sum \sigma_i$ so that

$$\sum_i D_X(q;\nu_i) = 0 \tag{7}$$

where ν_i is the p-vector of σ_i. Suppose $q \in \sigma_i$. Let σ_κ be σ be contracted toward q by a factor κ, say $\partial\sigma_\kappa = \sum \sigma_{\kappa_i}$. Note that

$$|\sigma_\kappa|_{-1,\alpha} = \kappa^{p+\alpha}|\sigma|_{-1,\alpha}, |\sigma_{\kappa_i}|_0 = \kappa^p|\sigma_i|_0.$$

Let ν_{κ_i} be the p-vector of σ_{κ_i} and γ_i its p-direction. If δ is the diameter of σ, we have

$$\begin{aligned}\left| \sum_i D_X(q;\nu_{\kappa_i}) - \int_{\partial\sigma_\kappa} D_X \right| &= \left| \sum_i \int_{\sigma_{\kappa_i}} (D_X(q;\gamma_i) - D_X(q';\gamma_i))dq' \right| \\ &\le \|D_X\|_{C^\alpha}(\kappa\delta)^\alpha \sum_i |\sigma_{\kappa_i}|_0 \\ &= \kappa^{p+\alpha}C\delta^\alpha|\partial\sigma|_0.\end{aligned}$$

Now

$$\begin{aligned} |X \cdot \partial\sigma_\kappa| &\le C|\partial\sigma_k|_\alpha \\ &\le C|\sigma_\kappa|_{-1,\alpha} \\ &\le C\kappa^{p+\alpha}|\sigma|_{-1,\alpha}. \end{aligned}$$

Therefore, by the definition of the Riemann integral

$$\begin{aligned} \left|\int_{\partial\sigma_\kappa} D_X\right| &= \left|\sum_i \int_{\sigma_{\kappa_i}} D_X(q';\gamma_i)dq'\right| = |\sum_i X \cdot \sigma_{\kappa_i}| \\ &= |X \cdot \partial\sigma_\kappa| \le C\kappa^{p+\alpha}|\sigma|_{-1,\alpha}. \end{aligned}$$

Combining these inequalities yields

$$\left|\sum_i D_X(q;\nu_i)\right| \le \kappa^{-p}\sum_i |D_X(q;\nu_{\kappa_i})| \le \kappa^\alpha C\delta^\alpha|\partial\sigma|_0 + \kappa^\alpha C|\sigma|_{-1,\alpha}.$$

Since κ is arbitrary, equation (7) follows. Lemma 2.1 shows there is a unique p-covector $D_X(q)$ for each q such that $D_X(q)\cdot\nu = D_X(q;\nu)$ for simple ν.

Linearity follows directly from the definition of D_X :

$$D_{X+Y} = D_X + D_Y; D_{tX} = tD_X.$$

Since $\int_{\sigma^{s+1}} D_X = X\cdot\sigma^{s+1}$ for all $s \ge -1$, it follows from Lemma 3.2 of [H3] that $\|D_X\|_{s,\beta} = \|X\|_{s,\beta}$ and $\|D_X\|^d_{s,\beta} = \|dX\|_{s,\beta}$ for all $-1 < (s,\beta) \le (r,\alpha)$. Therefore by Lemma 3.3 and Theorem 3.5 of [H3] we have $|D_X|_{s,\beta} = |X|_{s,\beta}$. Using Theorems 4.5 and 4.9 of [H3] we conclude

$$\|D_X\|_{C^{r,\alpha}} \le |X|_{r,\alpha} \le (p+1)\|D_X\|_{C^{r,\alpha}}.$$

Now let ω be a differential form satisfying $\|\omega\|_{C^{r,\alpha}} < C$. Set $X = \Psi(\omega)$. Once we show that $|X|_{r,\alpha} < \infty$ the proof is complete.

By Theorem 4.5 of [H3]

$$|\omega|_{r,\alpha} \le (p+1)\|\omega\|_{C^{r,\alpha}}.$$

We are reduced to showing $|X|_{r,\alpha} = |\omega|_{r,\alpha}$, but this follows as in the preceding discussion since $\int_{\sigma^{s+1}}\omega = X\cdot\sigma^{s+1}$ for all $s \ge -1$.

□

Henceforth, we let D_X denote the differential form associated to X as defined above.

Theorem 2.3. *Let* $X \in (\mathcal{A}^{1,\alpha})^*$. *Then*

$$D_{dX} = dD_X.$$

Proof. Since $dX \in (\mathcal{A}^\alpha)^*$, we may apply the preceding result to deduce the differential form D_{dX} is uniquely determined and is of class C^α.

Now

$$\int_{\partial\sigma} D_X = X\cdot\partial\sigma = dX\cdot\sigma = \int_\sigma D_{dX}.$$

Therefore D_X is regular.

By Stokes' theorem for regular forms,

$$dX \cdot \sigma = X \cdot \partial\sigma = \int_{\partial\sigma} D_X = \int_\sigma dD_X.$$

Since dD_X is continuous, if β is a p-direction and $q \in \mathbb{R}^n$ then

$$D_{dX}(q;\nu) = \lim_{i\to\infty} \frac{dX \cdot \sigma_i}{|\sigma_i|_0} = \lim_{i\to\infty} \frac{1}{|\sigma_i|_0} \int_{\sigma_i} dD_X(p;\nu)dp = dD_X(q;\nu).$$

Therefore

$$D_{dX} = dD_X.$$

□

The converse to Theorem 2.2 was proved in [H3] (Lemma 4.1 and Theorems 4.5 and 4.9). That is, $\omega \in \mathcal{B}^{r,\alpha}$ then $\Psi(\omega) \in (\mathcal{A}^{r,\alpha})^*$ and

$$\|\omega\|_{C^{r,\alpha}} \le |\Psi(\omega)|_{r,\alpha} \le (p+1)\|\omega\|_{C^{r,\alpha}}. \tag{8}$$

We summarize our results so far about the integration mapping Ψ from this and Theorems 2.2 and 2.3, taking note of the dimension of the chains.

Theorem A. *Fix $(r,\alpha) > 0$. The chain map Ψ is an isomorphism of chain complexes*

$$\begin{array}{ccccc} \left(\mathcal{A}_p^{r,\alpha}\right)^* & \xrightarrow{d} & \left(\mathcal{A}_{p+1}^{r-1,\alpha}\right)^* \cdots & \xrightarrow{d} & \left(\mathcal{A}_{p+r}^{\alpha}\right)^* \\ \Psi \quad \cong \uparrow & & \cong \uparrow & & \cong \uparrow \\ \mathcal{B}_p^{r,\alpha} & \xrightarrow{d} & \mathcal{B}_{p+1}^{r-1,\alpha} \cdots & \xrightarrow{d} & \mathcal{B}_{p+r}^{\alpha} \end{array}$$

The operator D is the inverse to Ψ and

$$\|\omega\|_{C^{r,\alpha}} \le |\Psi(\omega)|_{r,\alpha} \le (p+1)\|\omega\|_{C^{r,\alpha}}.$$

Corollary 2.4. *Two chainlets $A, B \in \mathcal{A}_p^{r,\alpha}$ are equal if and only if all integrals of forms $\omega \in \mathcal{B}^{r,\alpha}$ over them are the same*

$$\int_A \omega = \int_B \omega.$$

Two forms η and ν in $\mathcal{B}^{r,\alpha}$ are equal if and only iff all integrals of them over chainlets $A \in \mathcal{A}_p^{r,\alpha}$ are the same

$$\int_A \eta = \int_A \nu.$$

Proof. These follow from Theorem A. Let $A \in \mathcal{A}^{r,\alpha}$. Then

$$\begin{aligned} \textstyle\int_A \omega = 0 \quad \forall \omega \in \mathcal{B}^{r,\alpha} &\implies \Psi(\omega) \cdot A = 0 \quad \forall \omega \in \mathcal{B}^{r,\alpha} \\ &\implies X \cdot A = 0 \quad \forall X \in (\mathcal{A}_p^{r,\alpha})^* \\ &\implies A = 0 \text{ in } \mathcal{A}_p^{r,\alpha}. \end{aligned}$$

Let $\omega \in \mathcal{B}^{r,\alpha}$. Then $\Psi(\omega) \in (\mathcal{A}_p^{r,\alpha})^*$ and

$$\begin{aligned} \textstyle\int_A \omega = 0 \quad \forall A \in \mathcal{A}^{r,\alpha} &\implies \Psi(\omega) \cdot A = 0 \quad \forall A \in \mathcal{A}^{r,\alpha} \\ &\implies \Psi(\omega) = 0 \\ &\implies \omega = D_{\Psi(\omega)} = 0. \end{aligned}$$

□

For a polyhedral p-chain P in $\mathbb{R}^n$, define the $C^{r,\alpha}$ *operator norm* as

$$\|P\|_{C^{r,\alpha}} = sup \left\{ \left| \int_P \omega \right| : \omega \in \mathcal{B}^{r,\alpha}, \|\omega\|_{C^{r,\alpha}} \le 1 \right\}.$$

Corollary 2.5. *The $C^{r,\alpha}$ operator norm and $\mathcal{A}^{r,a}$ norm are equivalent. In particular,*

$$|P|_{r,\alpha} \le \|P\|_{C^{r,\alpha}} \le (p+1)|P|_{r,\alpha}$$

for all polyhedral chains P.

Proof. By Theorem A

$$\begin{aligned} |P|_{r,\alpha} = \sup_{X \ne 0} \tfrac{|X \cdot P|}{|X|_{r,\alpha}} &= \sup_{X \ne 0} \tfrac{|\int_P D_X|}{|D_X|_{r,\alpha}} \\ &= \sup_{\omega \ne 0} \left\{ \tfrac{|\int_P \omega|}{|\omega|_{r,\alpha}}, \omega \in \mathcal{B}^{r,\alpha} \right\}. \end{aligned}$$

The result follows from (8). □

Smoothing L^1 differential forms

We next investigate relations between the spaces of differential forms $\mathcal{J}^{r,\alpha}$ and the spaces of smooth forms $\mathcal{B}^{r,\alpha}$. If $\omega \in \mathcal{B}^{r,\alpha}$ then $\omega = D_{\Psi(\omega)}$.

Proposition 2.6. *Let ω be an L^1 differential form. If $\omega \in \mathcal{J}^{r,\alpha}$ for some $(r,\alpha) > 0$ then $D_{\Psi\omega} \in \mathcal{B}^{r,\alpha}, D_{\Psi\omega} = \omega$ a.e., and*

$$\|D_{\Psi\omega}\|_{C^{r,\alpha}} \le |\omega|_{r,\alpha} \le (p+1)\|D_{\Psi\omega}\|_{C^{r,\alpha}}.$$

Proof. If $\omega \in \mathcal{J}^{r,\alpha}$ then $\Psi(\omega) \in (\mathcal{A}^{r,\alpha})^*$ by Lemma 4.1 of [H3]. By Theorem 2.2 $D_{\Psi\omega} \in \mathcal{B}^{r,\alpha}$ and and

$$\|D_{\Psi(\omega)}\|_{C^{r,\alpha}} \le |\Psi(\omega)|_{r,\alpha} \le (p+1)\|D_{\Psi(\omega)}\|_{C^{r,\alpha}}.$$

But $|\Psi(\omega)|_{r,\alpha} = |\omega|_{r,\alpha}$. Since

$$\int_\sigma D_{\Psi\omega} = \Psi(\omega) \cdot \sigma = \int_\sigma \omega$$

for all simplexes σ, we conclude that $D_{\Psi\omega} = \omega$. a.e. □

For $\omega \in \mathcal{J}_p^{r+1,\alpha}$ we know $d\Psi(\omega) \in \left(\mathcal{A}_{p+1}^{r,\alpha}\right)^*$. Define

$$d\omega = D_{d\Psi(\omega)}.$$

Corollary 2.7. *If $\omega \in \mathcal{J}_p^{r+1,\alpha}$ then $d\omega \in \mathcal{B}_{p+1}^{r,\alpha}$,*

$$\|d\omega\|_{C^{r,\alpha}} \le |d\Psi(\omega)|_{r,\alpha} \le (p+1)\|d\omega\|_{C^{r,\alpha}}.$$

and

$$\Psi(d\omega) = d\Psi(\omega).$$

If $\omega \in \mathcal{B}_p^{r+1,\alpha}$ then $D_{d\Psi(\omega)}$ corresponds with the analytic definition of $d\omega$.

Proof. By Proposition 2.6 $d\omega \in \mathcal{B}_{p+1}^{r,\alpha}$. Now apply Theorem 2.2 to $d\Psi(\omega) \in \left(\mathcal{A}_{p+1}^{r,\alpha}\right)^*$ to deduce

$$\Psi(d\omega) = \Psi(D_{d\Psi(\omega)}) = d\Psi(\omega).$$

If $\omega \in \mathcal{B}_p^{r+1,\alpha}$ then $\Psi(\omega) \in \left(\mathcal{A}_p^{r+1,\alpha}\right)^*$ implying by Theorem 2.3 that

$$D_{d\Psi(\omega)} = dD_{\Psi(\omega)} = d\omega.$$

□

Geometric definition of the exterior derivative Corollary 2.7 gives us a geometric interpretation of the exterior derivative $d\omega$ of a differential form $\omega \in \mathcal{J}_p^{1,\alpha}$: If $q \in \mathbb{R}^n$ and β is a $(p+1)$-direction we have $d\omega(q;\beta) = D_{d\Psi(\omega)}(q;\beta)$. Therefore

$$d\omega(q;\beta) = \lim_{|\sigma_i|_0 \to 0} \frac{\int_{\partial\sigma_i}\omega}{|\sigma_i|_0}$$

where each σ_i is a simplex containing q, with p-direction the same as that of β. This shows that the exterior derivative extends the definition of the derivative of a smooth function of a real variable since

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{|I|_0\to 0}\frac{\int_{\partial I} f}{|I|_0}$$

where $I = [x, x+h]$.

Theorem B. *Fix $(r,\alpha) > 0$. The chain map Ψ is an epimorphism of chain complexes*

$$\begin{array}{ccccccc}
 & \left(\mathcal{A}_p^{r,\alpha}\right)^* & \xrightarrow{d} & \left(\mathcal{A}_{p+1}^{r-1,\alpha}\right)^* \cdots & \xrightarrow{d} & \left(\mathcal{A}_{p+r}^{\alpha}\right)^* \\
\Psi & \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\
 & \mathcal{J}_p^{r,\alpha} & \xrightarrow{d} & \mathcal{J}_{p+1}^{r-1,\alpha} \cdots & \xrightarrow{d} & \mathcal{J}_{p+r}^{\alpha}
\end{array}$$

For $\omega \in \mathcal{J}_p^{r,\alpha}$, $\omega = D_{\Psi(\omega)}$ a.e. and $|\Psi(\omega)|_{r,\alpha} = |\omega|_{r,\alpha}$. For $X \in \left(\mathcal{A}_p^{r,\alpha}\right)^*$ $X = \Psi(D_X)$. Furthermore, if ω is an L^1 form such that $\Psi(\omega) \in \left(\mathcal{A}_p^{r,\alpha}\right)^*$ then $\omega \in \mathcal{J}_p^{r,\alpha}$.

Proof. According to Lemma 4.2 of [H3] if ω is an L^1 form with $\Psi(\omega) \in \left(\mathcal{A}_p^{r,\alpha}\right)^*$ then $\omega \in \mathcal{J}_p^{r,\alpha}$ and $|\omega|_{r,\alpha} = |\Psi(\omega)|_{r,\alpha}$. The diagram commutes by Corollary 2.7. The remainder follows from Proposition 2.6. □

Theorems A and B highlight a special relationship between the spaces of differential forms $\mathcal{B}^{r,\alpha}$ and $\mathcal{J}^{r,\alpha}$. Namely,

$\mathcal{J}^{r,\alpha}$ is the unique largest subspace of L^1 differential forms containing $\mathcal{B}^{r,\alpha}$ such that $D_{\Psi\omega} \in \mathcal{B}^{r,\alpha}$ for all $\omega \in \mathcal{J}^{r,\alpha}$.

Two differential p-forms that are mapped under Ψ to the same co-chainlet must agree, except on a set with p-dimensional Hausdorff measure zero, in their domains of definition. But this is not a sufficient condition.

Example. *Let σ^0 be the positively oriented 1-simplex with support the unit interval $[0,1] \times \{0\}$ in $\mathbb{R}^2$. Set $\omega_1 = 0$ except on σ^0 where $\omega_1 = dx$. Then $\omega_1 \in \mathcal{J}^0$ since $\frac{\int_\sigma \omega_1}{|\sigma|_0} \le 1$ for all 1-simplexes σ. We see that $\omega_1 \notin \mathcal{J}^1$ by considering translations of σ^0 to create simple 1-dipoles $\sigma^1 = \sigma^0 - T_{\varepsilon v}\sigma^0$ where v is a fixed unit vector and $\varepsilon > 0$. Then $\frac{\int_{\sigma^1} \omega_1}{\|\sigma^1\|_1} = \frac{\int_{\sigma^1} \omega_1}{\varepsilon|\sigma^0|_0} = 1/\varepsilon$.*

In contrast, set $\omega_2 = 0$ except on the snowflake arc S where $\omega_2 = dx$. Then $\omega_2 \in \mathcal{J}^1$ since $\int_S \omega_2 = \lim \int_{P_k} \omega_2 = 0$ for any polyhedral approximators P_k.

Generalized Stokes' Theorem

THEOREM 2.8. *If ω is a differential form in $\mathcal{J}_p^{r+1,\alpha}$ and $A \in \mathcal{A}_{p+1}^{r,\alpha}$ then*

$$\int_A d\omega = \int_{\partial A} \omega.$$

Proof. If $\omega \in \mathcal{J}_p^{r+1,\alpha}$, then $d\omega = D_{d\Psi(\omega)}$. By Theorems A and B, $d\omega \in \mathcal{B}_{p+1}^{r,\alpha}$ and $\Psi(d\omega) \in \left(\mathcal{A}_{p+1}^{r,\alpha}\right)^*$. Also,

$$\Psi(d\omega) = d\Psi(\omega).$$

Hence $\int_A d\omega = \int_{\partial A} \omega$. □

This was proved in [H-N:1] and [H-N:2] for $\omega \in \mathcal{B}_{n-1}^{\alpha}, 0 < \alpha \le 1$ and codimension one domains. See [H2] and [H3] for domains of arbitrary codimension and $\omega \in \mathcal{B}_p^{r,1}$ with $r \ge 0$.

Relation to the L^1 norm Let f be a nonnegative measurable function defined on $[0,1]$, bounded by K. We have seen the graph of f supports a chainlet $\Gamma_f \in \mathcal{A}^1$. Its norm is estimated by the L^1 norm of f.

THEOREM 2.9. $|f|_{L^1}/K \le |\Gamma_f|_1 \le |f|_{L^1} + 1$.

Proof. The function f is approximated by step functions S_n with graphs Γ_n. Let $\Gamma_0 = I$. We have seen in [H3] that $\Gamma_f = \Gamma_0 + \sum_{i=1}^{\infty}(\Gamma_i - \Gamma_{i-1})$. These graphs may be decomposed into dipoles and the dipole mass of $\Gamma_i - \Gamma_{i-1}$ is the same as the L^1 norm of $S_i - S_{i-1}$. Thus

$$|\Gamma_f|_1 \le 1 + |f|_{L^1}.$$

We use Corollary 2.5 and the generalized Stokes' theorem to obtain the other inequality. We know that $|f|_{L^1} = \int_{\Gamma_f} y dx$. Since $\|y dx\|_{C^{Lip}} \le K$ we deduce $|\Gamma_f|_1 \ge |f|_{L^1}/K$. □

We have seen that the L^1 norm of f is the same as the integral $\int_{\Gamma_f} y dx$. One can integrate any smooth differential form over the graph. In particular for any $p > 0$, we have

$$\int_{\Gamma_f} y^p dx = \int_{\Gamma_{f^p}} y dx.$$

Thus

$$|f|_{L^p} = \left(\int_{\Gamma_f} y^p dx \right)^{1/p}.$$

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