

FLUX ACROSS NONSMOOTH BOUNDARIES AND THE FRACTAL DIVERGENCE THEOREM

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ABSTRACT. Replacing parametrization of a domain with polyhedral approximations we give an optimal extension of the Divergence Theorem. Permitted domains of integration, called *chainlets*, range from smooth submanifolds to structures that may not be locally Euclidean and have no tangent vectors defined anywhere. One may still calculate divergence over the domain, and flux through its boundary which itself may have no normal vectors defined anywhere.

INTRODUCTION

The real numbers \mathbb{R} are a completion of the rationals via the Euclidean metric. Continuity properties of real numbers, relative to the Euclidean metric, are at the heart of real analysis. Similarly, one may consider the vector space of p -dimensional simplicial chains $\sum_{i=1}^k a_i \sigma_i$ in \mathbb{R}^n and their completion w.r.t. a norm*. The Banach space obtained on completion has limit points that can be written as conditionally convergent series of simplicial chains,

$$A = \sum_{i=1}^{\infty} a_i \sigma_i,$$

called *chainlets*. In [H1] the author defined a family of norms giving geometric meaning to these infinite series of weighted simplexes and thus to chainlets. (See §1, below.)

Date: November 14, 1998.

1991 Mathematics Subject Classification. Primary: 06B10; Secondary: 06B40.

Key words and phrases. fractal, differential form, chainlet, Hodge, Laplace, Stokes' theorem, flux, curl, divergence.

*One may also work with ambient spaces of Riemannian manifolds. (See [H3].)

The integral of a smooth form ω over a chainlet $\sum_{i=1}^{\infty} a_i \sigma_i$ is defined using term by term integration

$$\int_{\sum_{i=1}^{\infty} a_i \sigma_i} \omega = \sum_{i=1}^{\infty} \int_{a_i \sigma_i} \omega.$$

Examples of chainlets include smooth submanifolds, fractals, vector fields, Dirac delta masses, Cantor sets, and stable manifolds and the theory shows how they all fit together continuously into Banach spaces. Some examples are further described in §2 and §3. The author [H4] has shown that every current and distribution corresponds to a chainlet. This not only provides a large source of examples, but it can be used to show a number of generalizations of classical results are optimal. While distributions and currents are defined abstractly as linear functionals on functions and differential forms, resp., we emphasize that chainlets have concrete geometric definition.

The classical theory of differential manifolds relies heavily on results of linear algebra of tangent spaces. Much of the work involves taking partitions of unity or checking coherence in the overlap maps. These techniques are not necessary or even valid for the theory of chainlets which are not assumed to be locally Euclidean and thus may have no tangent spaces. Instead, one replaces linear algebra on tangent spaces with analysis on simplexes. Parametrization is replaced with simplicial approximation. Partitions of unity are replaced with algebraic sums of chains. The unit normal bundle of a smooth submanifold B is replaced with $*B$, the chainlet that is the geometric Hodge $*$ of B . (See §3.) Because of continuity of fundamental operators, results such as Stokes' theorem for simplexes carry immediately over to chainlets. Other important results of calculus, algebraic topology, differential topology, and measure theory extend to chainlets giving a common language for these theories.

1. DIPOLES AND NORMS

An *oriented p -simplex* in \mathbb{R}^n is the oriented convex hull of $p + 1$ points in \mathbb{R}^n . We assume all simplexes are oriented henceforth. A *simplicial chain* in \mathbb{R}^n is a formal sum of simplexes in \mathbb{R}^n with real coefficients. We may assume that integration of smooth forms is defined over simplexes and thus over simplicial chains, and that Stokes' theorem is valid for simplicial chains $\int_{\partial S} \omega = \int_S d\omega$. The mass of a simplicial chain $\sum_i a_i \sigma_i$ is simply the weighted sum of masses of the simplexes $|S|_0 = \sum_i |a_i| m(\sigma_i)$ where m denotes p -dimensional Lebesgue measure. If $v \in \mathbb{R}^n$ is a vector let $|v|$ denotes its length. If σ is a p -simplex in

\mathbb{R}^n and $v \in \mathbb{R}^n$, define $T_v\sigma$ as the translate of σ by v . Its orientation is naturally induced from the orientation given on σ .

Mass of simplicial chains does not naturally measure geometric continuity. For example, the simplicial chain $\sigma - T_{tv}\sigma$ has mass that is twice the mass of σ unless $t = 0$. This problem is partially circumvented with polyhedral chains.

Polyhedral chains are equivalence classes of simplicial chains satisfying

$$S \sim T \iff \int_S \omega = \int_T \omega$$

for all smooth ω . Write $A = [S]$ and define $\int_A \omega = \int_S \omega$. For example, $-\sigma$ is identified with the same simplex as σ but with the opposite orientation. This definition takes into account overlapping simplexes with the opposite sign. The region of overlap is canceled. Polyhedral chains have naturally defined mass

$$|A|_0 = \inf\{|S|_0 : A = [S]\}.$$

The mass of an n -dimensional polyhedral chain $A = [\sigma - T_{tv}\sigma]$ in \mathbb{R}^n is a continuous function of t . Because of cancellation of overlapping, oppositely oriented simplexes, the mass tends to zero as $t \rightarrow 0$.

Unfortunately, mass alone is not enough to conclude that a chain and its nearby translate are close to each other. Consider a 2-simplex σ in \mathbb{R}^3 and v a vector not in the plane of σ . Then σ and $T_{tv}\sigma$ are disjoint if $t \neq 0$. The mass of $[\sigma - T_{tv}\sigma]$ is again exactly twice the mass of σ , until $t = 0$, at which point the mass becomes 0. We need finer norms than mass to be sensitive to geometric continuity.

Dipoles. A *simple p -dimensional 0-dipole* in \mathbb{R}^n is defined to be a p -simplex σ^0 with diameter ≤ 1 . A *simple p -dimensional 1-dipole* is a p -chain of the form

$$\sigma^1 = \sigma^0 - T_{v_1}\sigma^0$$

where $|v_1| \leq 1$ and σ^0 is disjoint from $T_{v_1}\sigma^0$. We inductively define simple j -dipoles. Given a vector v_j with $|v_j| \leq 1$ and a simple $(j-1)$ -dipole σ^{j-1} disjoint from $T_{v_j}\sigma^{j-1}$, define the *simple p -dimensional j -dipole* σ^j as the simplicial p -chain

$$\sigma^j = \sigma^{j-1} - T_{v_j}\sigma^{j-1}.$$

Thus σ^j is generated by vectors v_1, \dots, v_j , each with norm ≤ 1 , and a simplex σ^0 , where all translations of σ^0 through the vectors v_i are disjoint.

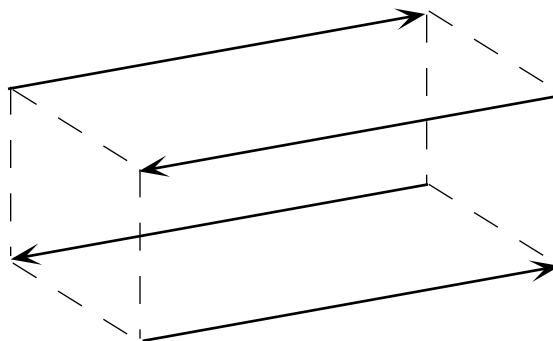


FIGURE 1. A simple 2-dipole, or quadrupole

For $j \geq 0$, a j -dipole in \mathbb{R}^n , is a polyhedral chain represented by a formal sum of simple j -dipoles, $D^j = \left[\sum_{i=1}^k a_i \sigma_i^j \right]$ with real coefficients a_i .

Dipole mass. Given a simple j -dipole σ^j , generated by a simplex σ^0 and constant vector fields v_1, \dots, v_j , with $|v_i| \leq |v_j| \leq 1, 1 \leq i \leq j$, define its j -dipole mass

$$\|\sigma^j\|_j = |\sigma^0|_0 |v_1| \cdots |v_j|.$$

For example, suppose σ^1 is a 1-dimensional 1-dipole, forming the oppositely oriented sides of a parallelogram. If each of these sides has length ε and the other sides each has length δ then the dipole mass $\|\sigma^1\|_1 = \varepsilon\delta$, regardless of the angle formed by the parallelogram. Even if the parallelogram is degenerate, the dipole mass is the same.

For j -dipoles $D^j, j \geq 0$, define j -dipole mass as

$$\|D^j\|_j = \inf \left\{ \sum_{i=1}^k |a_i| \|\sigma_i^j\|_j \right\},$$

where the infimum is taken over all representatives $D^j = \left[\sum_{i=1}^k a_i \sigma_i^j \right]$. In particular, $\|D^0\|_0 = |D^0|_0$.

r -norms. Let A be a polyhedral chain in \mathbb{R}^n and $r \in \mathbb{Z}, r \geq 1$. Define

$$(1) \quad |A|_r = \inf \left\{ \sum_{s=0}^r \|D^s\|_s + |C|_{r-1} \right\}$$

where the infimum is taken over all dipole decompositions

$$A = \sum_{s=0}^r D^s + \partial C.$$

These norms in this form were introduced in [H5], although earlier versions appeared in [H0]. We denote the Banach space of p -dimensional polyhedral chains completed with the r -norm by \mathcal{A}_p^r .

LEMMA 1.1. *If A is polyhedral, $|A|_s \leq |A|_r$ for all $r \leq s$.*

Proof. This follows directly from the definitions of the norms. \square

In [H6] the author shows that the s -norm of a chainlet in \mathcal{A}_p^r is well-defined by taking suitable limits of polyhedral chains and lower semi-continuous. This implies that the Banach spaces \mathcal{A}_p^r are nested and get larger and larger, including more and more strange and pathological limit points as r increases. For example, we see in §2 that the Dirac delta function is represented by a chainlet in \mathcal{A}_1^1 , its r th derivative (in the sense of distributions) by a chainlet in \mathcal{A}_1^{r+1} .

For $r \in \mathbb{Z}^+$, let $\mathcal{B}_p^{r,Lip}$ denote the real linear space of p -forms in \mathbb{R}^n with bounded norm $\|\omega\|_{C^{r,Lip}}$. That is, the r derivatives of each component function of ω exist, have uniformly bounded sup norm and satisfy a uniform Lipschitz condition.

Integration over chainlets.

THEOREM 1.2. *For A a polyhedral p -chain in \mathbb{R}^n and $\omega \in \mathcal{B}_p^{r-1,Lip}$ then*

$$\left| \int_A \omega \right| \leq (p+1) \|\omega\|_{C^{r-1,Lip}} |A|_r.$$

This is proved in [H5].

The integral of a form $\omega \in \mathcal{B}_p^{r-1,Lip}$ over a chainlet in \mathcal{A}_p^r . A in \mathbb{R}^n is defined by taking limits. If $A_k \rightarrow A$ are polyhedral chains in \mathbb{R}^n converging to A in the r -norm, define

$$\int_A \omega = \lim_{k \rightarrow \infty} \int_{A_k} \omega.$$

This is well-defined because of Theorem 1.2. This is equivalent[†] to the alternate definition given in the introduction. If $A = \sum_{i=1}^{\infty} a_i \sigma_i$ then

$$\int_A \omega = \sum_{i=1}^{\infty} \int_{a_i \sigma_i} \omega.$$

Another consequence of Theorem 1.2 is that $|\cdot|_r$ is a norm. Suppose $A \neq 0$ is a polyhedral chain. Then there exists an ∞ -smooth form ω such that $\int_A \omega \neq 0$. Thus

$$0 < \left| \int_A \omega \right| \leq (p+1) \|\omega\|_{C^{r-1}, Lip} |A|_r.$$

Hence $|A|_r \neq 0$.

2. EXAMPLES

We have chosen four far-ranging examples to illustrate chainlets.

Support of a chainlet The *support* of a polyhedral chain A is a closed set $spt(A)$ defined as follows: $x \in \mathbb{R}^n \setminus spt(A)$ iff there exists a neighborhood U of x in \mathbb{R}^n such that if ω is any smooth form supported in U then $\int_A \omega = 0$. The *support* $spt(A)$ of a chainlet $A \in \mathcal{A}_p^r$ is the set of points $q \in \mathbb{R}^n$ such that for every $\varepsilon > 0$ there exists a differential form $\omega \in \mathcal{B}^r$ such that $\int_A \omega \neq 0$ and $\omega(p) = 0$ outside $B_\varepsilon(q)$, the ball of radius ε about q .

It is important to keep in mind that there is much more to a chainlet than the subset of \mathbb{R}^n that forms its support. We will see that there may be many chainlets supported in a given set. For a trivial example, consider a positively oriented two simplex σ in \mathbb{R}^2 . The chains $\lambda\sigma$, $\lambda \in \mathbb{R}$, are distinct chains, with the same support. A more interesting example is the solenoid, seen below, which naturally supports quite different chainlets.

1. **Van Koch Snowflake** One may write the snowflake arc S as a sum of simplicial chains $\sum_{k=0}^{\infty} S_k$ where for $k \geq 1$, S_k is the sum of 4^k boundaries of triangles σ_k each of side length 3^{-k} . We show this series converges w.r.t. to the 1-norm: The partial sums satisfy $S_k + \cdots + S_j = \partial(\sigma_k + \cdots + \sigma_j)$. Thus

$$|S_k + \cdots + S_j|_1 \leq |\sigma_k|_0 + \cdots + |\sigma_j|_0 < 4^k / 3^{2k}.$$

[†]It is worth noting that if an infinite series is conditionally convergent w.r.t. a given norm then the sequence of partial sums converges w.r.t. the norm. Conversely, if $x_n \rightarrow x$ is a sequence converging w.r.t. a norm, then the infinite series $x_0 + \sum_{k=1}^{\infty} x_k - x_{k-1}$ conditionally converges w.r.t. the norm.

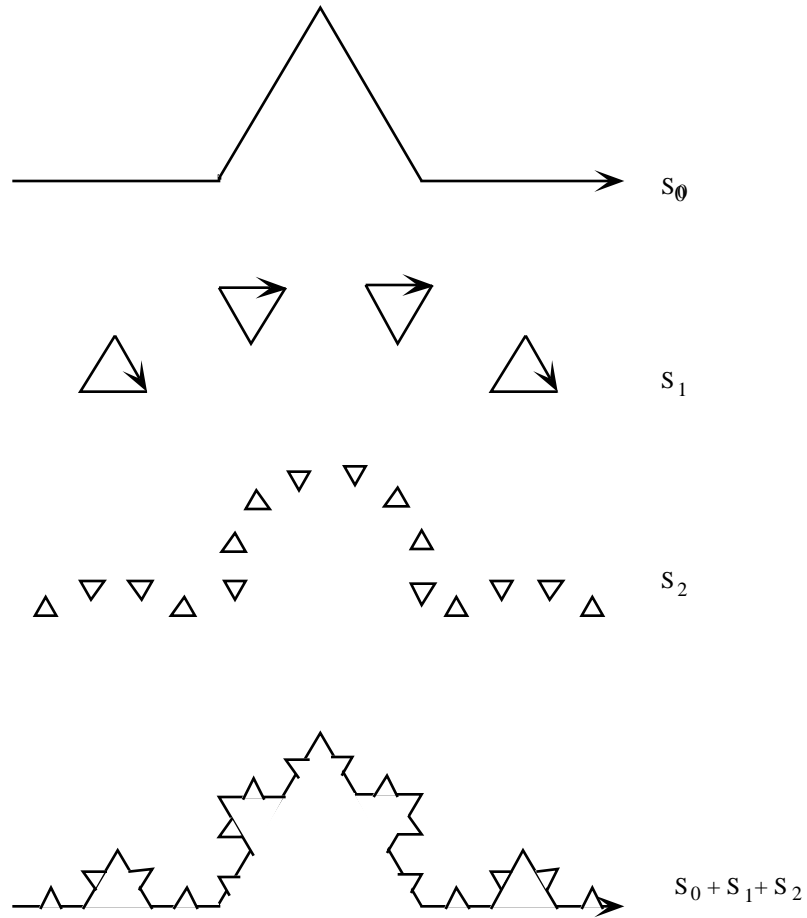


FIGURE 2. The snowflake as a sum of simplexes

Since the r.h.s. tends to 0 as $k, j \rightarrow \infty$, we know the infinite sum S is a well defined chainlet. We conclude that the snowflake is a current and we may integrate Lipschitz differential forms over it.

2. **Dirac delta function and its derivatives** We work in dimension one for simplicity of notation, but the construction can be extended to any dimension. Fix $p \in \mathbb{R}^1$. For each $k \geq 0$, let Q_k be a positively oriented interval with length 2^{-k} and centered at p . We claim that the sequence of polyhedral chains $D_k = 2^k Q_k$ converges w.r.t. the 1-norm. Notice that the mass of each chain is one. It suffices to estimate $|D_k - D_{k+1}|_1$. We show the difference $D_k - D_{k+1}$ is a 1-dimensional 1-dipole, a sum of 4 weighted simple dipoles. Divide D_k into 2 intervals Q_k with disjoint interiors, of length $2^{-(k+1)}$ and weighted by 2^k . Now D_{k+1} can also be written

as the sum of 2 intervals P_k of length $2^{-(k+1)}$ and weight 2^k , but the line segments are identical to each other. Since the distance between the line segments of P_k and those of Q_k is less than 2^{-k} we deduce

$$|D_k - D_{k+1}|_1 \leq 22^{-k}2^k2^{-(k+1)} = 2^{-k}.$$

We conclude that the sequence D_k is Cauchy in the 1-norm and its limit D has support p . The limit is canonically associated to the Dirac delta function. Since $D \in \mathcal{A}_1^1$, we may integrate smooth 1-forms ϕdx over it. Hence

$$\int_D \phi dx = \lim_{k \rightarrow \infty} \int_{D_k} \phi dx = \phi(p) = \delta(\phi).$$

The derivative of the Dirac delta function can also be realized geometrically, but as a chainlet $B \in \mathcal{A}_1^2$. One considers the quadrupoles (or 2-dipoles) formed by small oppositely oriented intervals centered at the endpoints of the D_k . It is left as an exercise to show that $\int_B \phi dx = \delta'(\phi)$.

3. **Toral solenoid** Let T be the 2-torus in \mathbb{R}^3 and $f : T \rightarrow T$ a smooth hyperbolic mapping that contracts the torus in one direction, expands it in the other and then wraps the torus around inside itself twice. The solenoid is defined as the intersection $\bigcap_{n=1}^{\infty} f^n T$. It is a set of points that supports many chainlets. For example, let Q be the solid torus positively oriented and $A_0 = Q/|Q|_0$. For $k \geq 0$, let $A_{k+1} = f(A_k)/|f(A_k)|_0$. Since the mass stays constant, the analysis here is similar to that for the Dirac delta function and one can use dipoles to show that A_k converges to a nonzero chainlet in \mathcal{A}_3^1 with support the solenoid. One can also find chainlets in \mathcal{A}_1^1 with support the solenoid as follows. Let B_0 be the oriented core circle in the torus which is not null homotopic. For $k \geq 0$, let $B_{k+1} = f(B_k)/|f(B_k)|_0$. Then B_k forms a Cauchy sequence in \mathcal{A}_1^1 and thus converges to a chainlet $B \in \mathcal{A}_p^1$. It is also possible to find chainlets in \mathcal{A}_0^1 with support the solenoid by choosing a countable dense subset and forming a Dirac mass at each of these points so that their total mass is finite. In the next section we find a chainlet in \mathcal{A}_2^1 with support the solenoid.
4. **Graphs of L^1 functions** The graph Γ of a nonnegative L^1 function over an interval $[a, b]$ supports a chainlet. One merely approximates Γ with graphs of monotone increasing step functions Γ_n . These Γ_n are naturally oriented to form simplicial chains and these form a Cauchy sequence in the 1-norm. The difference $\Gamma_n - \Gamma_{n+1}$ is a dipole and so its 1-norm is bounded above by the area between

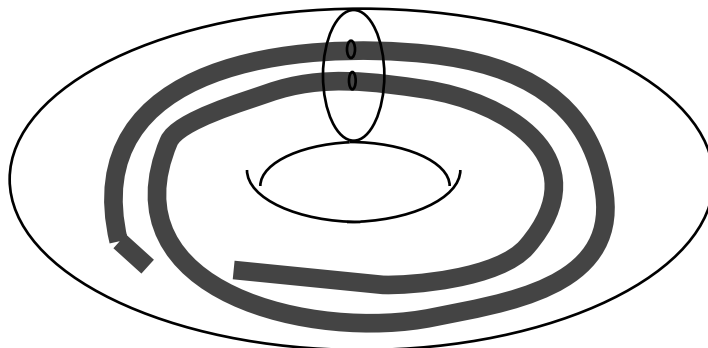


FIGURE 3. Toral solenoid

the two graphs Γ_n and Γ_m . Thus $\{\Gamma_n\}$ forms a Cauchy sequence in the 1-norm, converging to a chainlet Γ whose support is in the graph of f . One can think of Γ as the x -component of the graph of f .

3. DIV, GRAD, AND CURL FOR FRACTALS

Banach spaces of chainlets have standard operators defined on them. In this paper we consider the boundary, pushforward, and geometric Hodge * operators. Each is first defined for simplexes and extended to simplicial chains by linearity. Differential forms are used to prove the operators are well-defined on polyhedral chains. Finally, the operators are proved to be continuous w.r.t. the norms, showing they are defined on chainlets. In practice, most of the work comes in establishing the first and last steps. For each operator there is a duality theorem relating chainlets to differential forms. We demonstrate this method of proof for the boundary operator.

3.1. Boundary operator. The boundary of a simplicial chain is defined in the standard way. If $S \sim T$ are simplicial chains we apply Stokes' theorem for simplicial chains to deduce

$$\int_{\partial S} \omega = \int_S d\omega = \int_T d\omega = \int_{\partial T} \omega.$$

Hence $\partial S \sim \partial T$, implying that the boundary operator is well-defined on polyhedral chains. The boundary operator on polyhedral chains is bounded w.r.t. the r -norms. That is,

LEMMA 3.1.

$$|\partial A|_{r+1} \leq |A|_r.$$

Proof. This follows immediately from the definition of the r -norms. \square

We may thus define the boundary ∂A of an chainlet $A \in \mathcal{A}_p^r$. In particular, the **boundary operator**

$$\partial : \mathcal{A}_p^r \longrightarrow \mathcal{A}_{p-1}^{r+1}$$

is defined for $r \geq 0$. It restricts to the usual boundary operator on polyhedral chains and satisfies Stokes' theorem. The boundary operator is dual to the exterior derivative of forms, leading to Stoke's theorem for chainlets.

THEOREM 3.2. Generalized Stokes' theorem *Let $r \geq 0$. If A is an r -generalized p -chain and ω is a differential $(p-1)$ -form of class \mathcal{B}^{r+1} then ∂A is a $(r+1)$ -generalized $(p-1)$ -chain, $d\omega$ is differential p -form of class \mathcal{B}^r and*

$$\int_A d\omega = \int_{\partial A} \omega.$$

Proof. By Stokes' theorem for simplexes, continuity of the boundary operator and of the integral,

$$\begin{aligned} \int_A d\omega &= \int_{\sum_{i=1}^{\infty} a_i \sigma_i} d\omega = \sum_{i=1}^{\infty} a_i \int_{\sigma_i} d\omega = \sum_{i=1}^{\infty} a_i \int_{\partial \sigma_i} \omega \\ &= \int_{\sum_{i=1}^{\infty} a_i \partial \sigma_i} \omega \\ &= \int_{\partial \sum_{i=1}^{\infty} a_i \sigma_i} \omega \\ &= \int_{\partial A} \omega. \end{aligned}$$

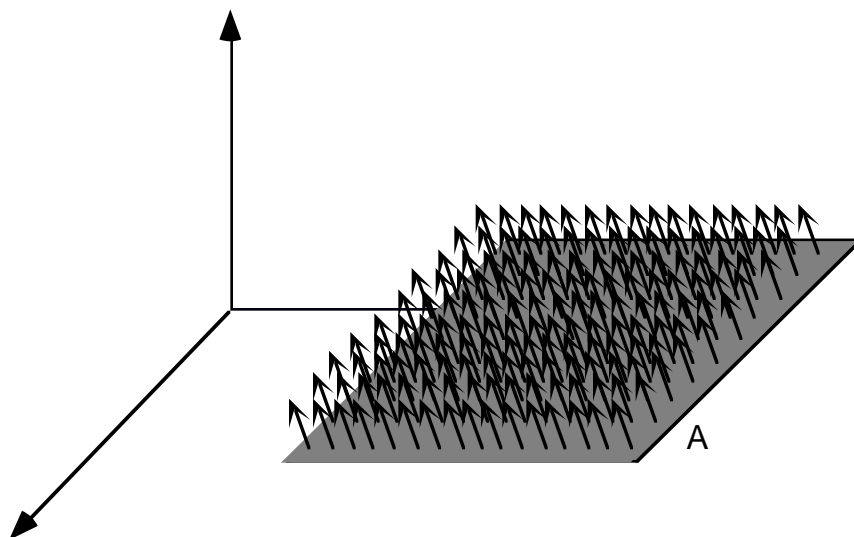
\square

In [H4] it is shown that this generalization of Stokes' theorem is optimal for integrands of smooth forms. That is, all possible domains of integration arise as chainlets. For 1-dimensional chains A Stokes' theorem implies that the Fundamental Theorem of Calculus is valid for all r -generalized 1-chains, e.g., fractal arcs. Here, ω is taken to be a function f and df is its gradient. For arcs A with endpoints p and q , this is usually written $\int_p^q f(x)dx = \int_A df$.

3.2. Pushforward operator. If f is a mapping of class \mathcal{B}^{r+1} , $r \geq 0$, one can define the pushforward operator or change of variables operator $f_{\#} : \mathcal{A}_p^r \rightarrow \mathcal{A}_p^r$. The pushforward operator on chainlets is dual to the pullback operator on forms leading to a *change of variables* theorem for chainlets.

THEOREM 3.3. *Let $r \geq 0$. If A is an r -generalized p -chain, and $\omega \in \mathcal{B}^r$ and $f \in \mathcal{B}^{r+1}$ then*

$$\int_{f_{\#}A} \omega = \int_A f^{\#}\omega.$$

FIGURE 4. A polyhedral approximation to $*R$

(See [H1] for more details.)

Several new operators are defined on chainlets. The main one we discuss here is the geometric Hodge $*$ operator $*$. This, along with the boundary operator, leads to optimal Green and Gauss theorems for chainlets. Combinations and modifications of these operators leads to geometric Laplace operators, Lie derivative, Dirac operators, and coboundary operators on chainlets.

3.3. Geometric Hodge star operator. $*$: $\mathcal{A}_p^r \longrightarrow \mathcal{A}_{n-p}^r$ is defined in ([H3]) for $r > 0$. To give the idea, Figure 3.2 illustrates a polyhedral approximation to $*R$ where R is the oriented rectangle depicted in \mathbb{R}^3 . $*R$ is found by taking a limit in the 1-norm of similar sums of tiny equally spaced 1-simplexes, orthogonal to R and whose total length for each sum is the same as the area of R . Even in this simplest example of a rectangle $*R$ is not locally Euclidean, showing that fractal-like structures are naturally associated to smooth ones. (See Figure 3.2.)

THEOREM 3.4. Generalized Hodge star theorem *If A is an r -generalized p -chain, and $\omega \in \mathcal{B}^r$ then $*A$ is a chainlet in \mathcal{A}_p^r , $*\omega \in \mathcal{B}^r$ and*

$$\int_A *\omega = \int_{*A} \omega.$$

For the proof, see [H3]. Using a combinatorial definition, others have defined a local dual to the Hodge star operator, but the integral equation of Theorem 3.4 does not hold.

The boundary and geometric Hodge star operators lead to generalized Green and Gauss theorems. Let S be a smooth, oriented surface with boundary in \mathbb{R}^3 . The usual way to integrate the curl of a vector field X over S is to integrate the dot product of $\text{curl}X$ with the unit normal vector field to S . According to Green's Theorem, this quantity equals the integral of X over ∂S . We have already seen we do not require the existence of tangents to ∂S to calculate the integral of X over it. By working with the differential 1-form ω associated to X via Euclidean coordinates and the geometric Hodge $*$ operator $*$, we no longer require the existence of any unit normals to S to integrate $\text{curl}X$ over A . Instead we use the geometric Hodge star operator applied to A , which is always defined if A is a chainlet. If ω is the differential 1-form representing the vector field X it is well known that $*d\omega$ represents $\text{curl}X$. We see now that

$$\int_{*S} *d\omega \text{ represents the integral of the } \text{curl}X \text{ over } S.$$

THEOREM 3.5. Fractal Green's theorem *If S is an chainlet in \mathcal{A}_p^r and $\omega \in \mathcal{B}^{r+1}$ then ∂S is a $(r+1)$ -generalized $(p-1)$ -chain, $*S$ is r -generalized, $*d\omega \in \mathcal{B}^r$ and*

$$\int_{\partial S} \omega = \int_{*S} *d\omega.$$

Proof. By Theorems 3.2 and 3.4 $\int_{\partial S} \omega = \int_S d\omega = \int_{*S} *d\omega$. □

For smooth surfaces with smooth boundary, we have $\int_{\partial S} X \cdot nds = \int_{\partial S} \omega$. Since $*d\omega$ corresponds to $\text{curl}X$. It follows that $\int_{*S} *d\omega = \int_S \text{curl}X \cdot ndA$. Therefore the preceding theorem generalizes and simplifies Green's theorem: $\int_S \text{curl}X \cdot ndA = \int_{\partial S} X \cdot nds$.

If A is a chainlet then $*\partial A$ plays the role of a normal vector field on its boundary, even though the boundary of A may have no normal vectors defined. Thus one may calculate flux across fractal boundaries and obtain a fractal divergence theorem.

The usual way to calculate flux of a vector field X across a boundary of a smooth solid region D in space is to integrate the dot product of X with the unit normal vector field to ∂D over the domain ∂D . According to Gauss' Divergence Theorem, this quantity equals the integral of the divergence of X over D . *By working with the differential form ω associated to X via Euclidean coordinates and the operator $*$, we no*

longer require the existence of any unit normals to ∂D to calculate the flux of X across ∂D . We see that

$$\int_{*\partial D} \omega \text{ represents the flux of } X \text{ across } \partial D.$$

THEOREM 3.6. Fractal divergence theorem *If D is an r -generalized p -chain and ω is of class \mathcal{B}^{r+1} then*

$$\int_{*\partial D} \omega = \int_D d * \omega.$$

Proof. $\int_{*\partial D} \omega = \int_{\partial D} * \omega = \int_D d * \omega.$ □

For smooth surfaces, we have $\int_S X \cdot ndA = \int_{*S} \omega$. It is well known that $d*\omega$ corresponds to $\text{div}X$. Therefore the preceding theorem generalizes and simplifies the Divergence Theorem of Gauss: $\int_{\partial D} X \cdot ndA = \int_D \text{div}X dV$.

Some authors have defined the integral over fractal boundaries using the integral of the derived form over the interior, i.e., using the generalized Stokes' theorem, as the definition. Instead, the integrals in the preceding five theorems are defined independently and are shown to satisfy the generalized Stoke's theorem.

3.4. Examples revisited.

1. **Van Koch snowflake** One may calculate flux of a vector field F across the snowflake S as $\int_{*S} \omega$ where ω is the 1-form determined by F using the Euclidean inner product.
2. **Dirac delta function and its derivatives** Distributions and their derivatives can be realized more systematically using the operator $*$ defined in §3 below. We say a distribution c is *associated* to a chainlet A if $c(\phi) = \int_A \phi dx$ for all test functions ϕ . In [H3] it is shown that if c is a distribution associated to the chainlet A then c' is associated to $*\partial A$.
3. **Toral solenoid** Recall the chainlet B in \mathcal{A}_1^1 found by iterating the core circle via the mapping f . supported in the solenoid. A 2-chainlet in \mathcal{A}_2^1 can be found by applying the $*$ operator to B . In some sense, this $*B$ acts as a normal bundle to B .
4. **Graph of an L^1 function** One may calculate flux of a vector field F across the x -component of the graph Γ of a nonnegative L^1 function as $\int_{*\Gamma} \omega$ where, again, ω is the 1-form determined by F . We give an important example. Let $F = ye_2$ where $\{e_1, e_2\}$ is the Euclidean basis of \mathbb{R}^2 . Then $\omega = ydy$ corresponds to F . Applying

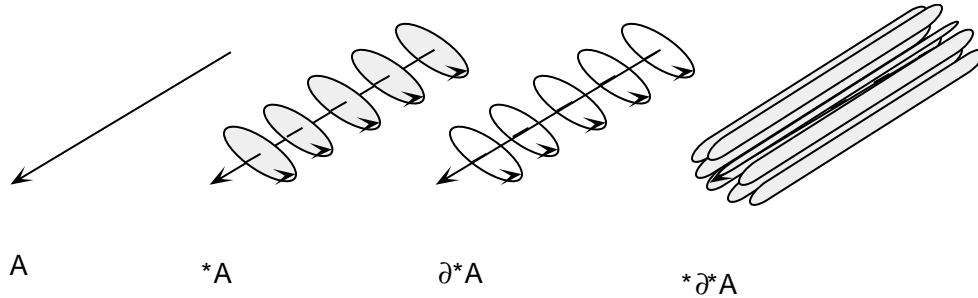


FIGURE 5. Polyhedral approximation to the coboundary of a line segment

Stokes' theorem we calculate the flux of F across Γ to be

$$\int_{*\Gamma} ydy = - \int_{\Gamma} ydx = \int_S dx dy = \int_0^1 f(x)dx.$$

Here, S denotes the subgraph of f . This links Lebesgue theory to chainlets.

Geometric Laplace operator The geometric Hodge $*$ operator $*$ leads immediately to definitions of the geometric coboundary operator δ on chainlets, defined as $\delta = (-1)^{n(p+1)+1} * \partial *$ (see figure 3), and the geometric Laplace operator $\Delta = \delta \partial + \partial \delta$. If A is an r -generalized p -chain, then δA is an $(r + 1)$ -generalized $(p + 1)$ -dimensional chain and ΔA is an $(r + 2)$ -generalized p -dimensional chain.

It follows readily from Theorems 3.4 and 3.2 that for $\omega \in \mathcal{B}^{r+1}$ and A an r -generalized p -chain

$$\int_A \delta \omega = \int_{\delta A} \omega.$$

and for $\omega \in \mathcal{B}^{r+2}$ then

$$\int_A \Delta \omega = \int_{\Delta A} \omega.$$

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