A GENERALIZATION OF THE DUAL ELLENTUCK THEOREM

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Abstract.- We prove versions of the Dual Ramsey Theorem and the Dual Ellentuck Theorem for families of partitions which are defined in terms of games.

0. Introduction

The study of filters associated with the Dual Ramsey Theorem of Carlson and Simpson [2] was initiated in [13]. [13] was however somewhat limited in scope, as it took the Dual Ramsey Theorem for granted. The proof (by induction) of the theorem which is given in [2] involves heterogeneous objects (letters of fixed "alphabets"), and this was an obstacle to a straightforward rewriting of the proof in terms of filters. We now return to the subject with a key new tool, namely the notion of game.

Our approach goes back to Kastanas [12] who obtained a characterization of completely Ramsey sets in terms of games. The idea of using analogous games in the framework of Dual Ramsey Theory is not new, as it was considered in 1984 by Voigt and the second author (see [19]), who were looking for an alternative proof of Carlson's Lemma (Lemma 2.4 in [2]) which is the key to the Dual Ramsey Theorem and the other results of [2]. The attempt misfired and was in fact misguided. The point is that, just as in Baumgartner's proof of Hindman's Theorem [1], which can be seen as a model for it, Carlson's proof in [2] of his lemma can be easily reformulated in the language of games. Once this has been realized, it is a simple matter to isolate

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a combinatorial property of filters such that filters with this property (we call them game-filters) satisfy the appropriate relativization of Carlson's Lemma. One can then proceed and obtain versions for game-filters of the Dual Ramsey Theorem and the Dual Ellentuck Theorem. Game-filters are thus to the Dual Ramsey Theorem what Ramsey ultrafilters are to Ramsey's Theorem.

As their name indicates, game-filters are defined in terms of games. One would certainly expect that, just as Ramsey ultrafilters, they could also be characterized combinatorially in other terms. Such a further characterization has however so far escaped us, which has some unfortunate consequences. For one thing we are unable to show that game-filters are the only filters which verify the relativized versions of the Dual Ramsey Theorem and the Dual Ellentuck Theorem. In other words, our results lack converses. Moreover we were unable to determine whether the Continuum Hypothesis implies the existence of game-filters (that game-filters consistently exist has been shown in [6]).

We do not claim that as far as filters of partitions are concerned, the notion of game-filter is <u>the</u> right notion in the context of Dual Ramsey Theory. Carlson's Lemma appears in two guises in [2]. We already mentioned Lemma 2.4 which is the form we use in the present paper, but what Carlson actually proved is a strengthened, "specialized " version of it (Theorem 6.3 in [2] ; see also the main result of [9]). The filters of partitions that can be associated with the second form of Carlson's Lemma are very different from the ones we consider in this paper, since their members have only finitely many infinite blocks.

We follow the lead of Mathias who obtained in [17] a version of Ellentuck's Theorem for a class of filters which are not necessarily maximal (see also [15] and [5]). In fact we go one step further and relinquish any reference to the notion of filter, our main motivation being to make proofs more transparent by eliminating unnecessary hypotheses.

Our objects of study are called game-families. Their definition, again in terms of games, was inspired by Proposition 16.2 of [15]. Our main results in this context are generalizations of the Dual Ramsey Theorem and the Dual Ellentuck Theorem.

Section 1 presents basic definitions and notation concerning partitions. Section 2 describes the games which are used throughout the paper. Section 3 introduces the notion of game-family and gives some examples (some more examples are to be found in Section 10). Section 4 is devoted to our generalization of Carlson's Lemma. Our proof follows rather closely that of Carlson in [2] and makes essential use of the Hales-Jewett Theorem. The results of Section 5 can be viewed as the core of what could be called Dual Ellentuck Theory. In fact the generalizations of the Dual Ellentuck Theorem and the Dual Ramsey Theorem which are presented in Sections 6, 7 and 8 follow in a more or less straightforward manner from these results.

Section 6 presents a first generalization of the Dual Ellentuck Theorem. This one is formulated in topological terms and was inspired by a remark in [14]. The (easy) arguments behind Lemmas 6.2 and 6.3 come from [15]. Section 7 is devoted to a second generalization of the Dual Ellentuck Theorem, which is this time formulated in terms of partially ordered sets. Section 8 presents our generalization (and extension) of the Dual Ramsey Theorem. In Section 9 we consider two variants of Carlson's Lemma. Finally, Section 10 is concerned with existence and combinatorial properties of game-filters.

1. Partitions

In this section we introduce some notation.

Given two sets A and B, B^A denotes the set of all functions from A to B.

Let $\alpha, \beta \leq \omega$. By a *partition of α into β blocks* we mean an onto function $X : \alpha \to \beta$ with the property that $\min\big(X^{-1}(\{k\})\big) < \min\big(X^{-1}(\{m\})\big)$ for all $k, m \in \beta$ with $k < m$.

Thus the blocks of a partition are ordered as their leaders (i.e. their least elements).

$(\alpha)^\beta$ denotes the set of all partitions of α into β pieces.

We define the *block function* $\mathcal{B} : (\alpha)^\beta \to P\big(P(\alpha)\big)$ and the *leader function* $\ell : (\alpha)^\beta \times \beta \to \alpha$ by $\mathcal{B}(X) = \{X^{-1}(\{k\}) : k \in \beta\}$ and $\ell(X,k) = \min\big(X^{-1}(\{k\})\big)$.

So, $\mathcal{B}(X)$ is simply the set of blocks of X, and the function $k \longmapsto \ell(X,k)$ enumerates the leaders of X in increasing order.

The *segment function* s on $(\alpha)^\beta \times \beta$ is defined by $s(X,k) = X \restriction \big(1+\ell(X,k)\big)$.

Notice that $s(X,k)$ is a partition of $1+\ell(X,k)$ having $k+1$ blocks. The only block of $s(X,0)$ is $\{0\}$. For $k > 0$, the first k blocks of $s(X,k)$ are : $\{m \in X^{-1}(\{0\}) : m < \ell(X,k)\}$, $\{m \in X^{-1}(\{1\}) : m < \ell(X,k)\}, \ldots,$ $\{m \in X^{-1}(\{k-1\}) : m < \ell(X,k)\}$, and its last block is $\{\ell(X,k)\}$.

If X and Y are partitions of α, we say that Y is *coarser than* X, or that X *is finer than* Y, and we write $Y \leq X$, if each block of Y is a union of blocks of X.

Given $\gamma \leq \omega$ and a partition X of α, we put $(X)^\gamma = \{Y \in (\alpha)^\gamma : Y \leq X\}$.

That is, $(X)^\gamma$ is the set of all partitions of α into γ blocks that are coarser than X.

Given $X \in (\alpha)^\beta$, $k < \beta$ and $\gamma \le \omega$ such that $k+1+\gamma \le \beta$, we let

$$(k, X)^\gamma = \{Y \in (X)^{k+1+\gamma} : s(X,k) = s(Y,k)\}.$$

Hence, $(k, X)^\gamma$ is the set of all partitions of α into $(k+1)+\gamma$ blocks that are coarser than X but have the same $k+1$ first leaders as X.

If $X \in (\omega)^\omega$ and $k \in \omega$, we set $< X >_k = \{s(T, k+1) : T \in (k, X)^\omega\}$.

Thus, t is in $< X >_k$ exactly when for some $i \ge k+1$, t is a partition of $1+\ell(X,i)$ into $k+2$ blocks that is coarser than $X \restriction (1+\ell(X,i))$, and whose leaders are $\ell(X,0), \ell(X,1), \ldots, \ell(X,k), \ell(X,i)$.

If $X \in (\omega)^\omega$ and $k, n \in \omega$, we set $< X >_k^n = \{s(T, k+1+n) : T \in (k, X)^\omega\}$.

Note that $< X >_k^0 = < X >_k$.

Given a nonempty set J and $X_j \in (\omega)^\omega$ for $j \in J$, we let $\bigsqcap_{j \in J} X_j$ denote the finest partition of ω that is coarser than every X_j.

(See [13] for a detailed construction of $\bigsqcap_{j \in J} X_j$).

Let us finally introduce the following operations A (for amalgamation) and D (for disamalgamation).

For each $\delta \le \omega$, we let I_δ denote the identity function on $1+\delta$.

Thus $\mathcal{B}(I_i) = \{\{j\} : j \le i\}$ for every $i < \omega$.

Let $X \in (\omega)^\omega$ and $t \in (I_i)^m$, where $0 < m < i < \omega$. We define $A(t, X) \in (\omega)^\omega$ by letting

$$\mathcal{B}\big(A(t,X)\big) = \Big\{ \bigcup_{j \in t^{-1}(\{r\})} X^{-1}(\{j\}) : r < m \Big\} \bigcup \{X^{-1}(\{q\}) : q > i\}.$$

So if the m blocks of t are $e_0, \ldots, e_{m-1}$, the blocks of $A(t,X)$ are

$$\bigcup_{j \in e_0} X^{-1}(\{j\}), \ldots, \bigcup_{j \in e_{m-1}} X^{-1}(\{j\}), X^{-1}(\{i+1\}), X^{-1}(\{i+2\}), \ldots$$

For each $Y \in \big(m-1, A(t,X)\big)^\omega$, we define $D(t,X,Y) \in (\omega)^\omega$ by letting $\mathcal{B}\big(D(t,X,Y)\big) = E \cup H \cup K$, where

$$E = \bigcup_{r<m} \Big\{ X^{-1}(\{j\}) : j \in t^{-1}(\{r\}) - \{\ell(t,r)\} \Big\}$$

$$H = \Big\{ \big(Y^{-1}(\{r\})\big) - \Big(\bigcup_{j \in t^{-1}(\{r\}) - \{\ell(t,r)\}} X^{-1}(\{j\}) \Big) : r < m \Big\}$$

$$K = \{Y^{-1}(\{q\}) : q \ge m\}.$$

That is, $D(t,X,Y)$ is obtained from Y by "disamalgamating" $Y^{-1}(\{0\}),\ldots,Y^{-1}(\{m-1\})$ as follows. Let $r \leq m-1$ and set $t^{-1}(\{r\}) = \{j_0, j_1, \ldots, j_q\}$. Then

$$Y^{-1}(\{r\}) = B \cup X^{-1}(\{j_0\}) \cup X^{-1}(\{j_1\}) \cup \ldots \cup X^{-1}(\{j_q\})$$

for some $B \subseteq \omega$. We stipulate that

$$X^{-1}(\{j_0\}) \cup B, X^{-1}(\{j_1\}), \ldots, X^{-1}(\{j_q\})$$

will be blocks of $D(t,X,Y)$. So, the leaders of $D(t,X,Y)$ are

$$\ell(X,0), \ell(X,1), \ldots, \ell(X,i), \ell(Y,m), \ell(Y,m+1), \ldots$$

2. Games

We now introduce the games which will be used throughout the paper.

Let $\mathcal{F} \subseteq (\omega)^\omega, k \in \omega, X \in \mathcal{F}$ and $\mathcal{W} \subseteq \mathcal{F}^\omega$ be given. We define the two-person game $G_{\mathcal{F}}(k, X, \mathcal{W})$ as follows :

Each player makes ω moves. I starts by selecting $Y_0 \in \mathcal{F} \cap (k,X)^\omega$; II then chooses $Z_0 \in \mathcal{F} \cap (k, Y_0)^\omega$; I now picks $Y_1 \in \mathcal{F} \cap (k+1, Z_0)^\omega$; II answers by playing $Z_1 \in \mathcal{F} \cap (k+1, Y_1)^\omega$; I then selects $Y_2 \in \mathcal{F} \cap (k+2, Z_1)^\omega$; II answers with $Z_2 \in \mathcal{F} \cap (k+2, Y_2)^\omega$; etc.

Observe that

$$X \geq Y_0 \geq Z_0 \geq Y_1 \geq Z_1 \geq \ldots$$

Also, $Y_0, Z_0, Y_1, Z_1, \ldots$ have the same $k+1$ first leaders as X ; $Y_1, Z_1, Y_2, Z_2, \ldots$ have the same $k+2$ first leaders as Z_0 ; $Y_2, Z_2, Y_3, Z_3, \ldots$ have the same $k+3$ first leaders as Z_1 ; etc.

I is said to win if $(Z_0, Z_1, Z_2, \ldots) \in \mathcal{W}$.

Notice that $\bigsqcap_{i \in \omega} Z_i = \bigcup_{i \in \omega} s(Z_i, k+1+i)$.

Given $W \subseteq (\omega)^\omega$, we let $G_{\mathcal{F}}(k, X, W)$ stand for

$$G_{\mathcal{F}}(k, X, \{(Z_0, Z_1, \ldots) \in \mathcal{F}^\omega : \bigsqcap_{i \in \omega} Z_i \in W\}).$$

A standard argument yields the following.

Lemma 2.1.- *Let* $\mathcal{F} \subseteq (\omega)^\omega$ *and* $W_i \subseteq \mathcal{F}$ *for* $i \in \omega$*. Further let* $k \in \omega$ *and* $X \in \mathcal{F}$ *be such that* II *has no winning strategy in* $G_{\mathcal{F}}(k, X, \mathcal{W})$*, where* $\mathcal{W}$ *consists of all* $(Z_0, Z_1, \ldots) \in \mathcal{F}^\omega$ *such that* $Z_i \in W_i$ *for all* $i \in \omega$*. Then* I *has a winning strategy in* $G_{\mathcal{F}}(k, X, \mathcal{W})$*.*

Proof – We define a winning strategy σ for I in $G_{\mathcal{F}}(k, X, \mathcal{W})$ as follows. Let $\mathcal{W}_1$ be the set of all $(Z_1, Z_2, \ldots) \in \mathcal{F} \times \mathcal{F} \times \cdots$ such that $Z_i \in W_i$ for all $i \geq 1$. Then there exists $Y_0 \in \mathcal{F} \cap (k, X)^\omega$ such that for every $Z \in \mathcal{F} \cap (k, Y_0)^\omega, Z \in W_0$ and II has no winning strategy in $G_{\mathcal{F}}(k+1, Z, \mathcal{W}_1)$. We put $\sigma(\phi) = Y_0$.

Now let II play Z_0. Let $\mathcal{W}_2$ consist of all $(Z_2, Z_3, \ldots) \in \mathcal{F} \times \mathcal{F} \times \cdots$ such that $Z_i \in W_i$ for all $i \geq 2$. Then there exists $Y_1 \in \mathcal{F} \cap (k+1, Z_0)^\omega$ such that for every $Z \in \mathcal{F} \cap (k+1, Y_1)^\omega, Z \in W_1$ and II has no winning strategy in $G_{\mathcal{F}}(k+2, Z, \mathcal{W}_2)$. We put $\sigma(Z_0) = Y_1$. etc. ■

3. Game-families

This section is devoted to the notion of game-family. Let us start with the definition.

A *game-family* is a nonempty subset $\mathcal{F}$ of $(\omega)^\omega$ which satisfies the following two conditions :

(0) For every $X \in \mathcal{F}$, II has no winning strategy in $G_{\mathcal{F}}(0, X, \mathcal{F})$.

(1) Let $X, Y \in (\omega)^\omega$ and $i, j \in \omega$ be such that :

$$\mathcal{B}(Y) = \big(\mathcal{B}(X) - \{X^{-1}(\{i\}), X^{-1}(\{j\})\}\big) \bigcup \{X^{-1}(\{i\}) \cup X^{-1}(\{j\})\}.$$

Then $X \in \mathcal{F}$ if and only if $Y \in \mathcal{F}$.

Condition (1) is a type of closure under finite changes : if Y is obtained from $X \in (\omega)^\omega$ by amalgamating finitely many blocks of X, then $X \in \mathcal{F}$ iff $Y \in \mathcal{F}$.

As is shown by the following lemma, game-families satisfy a strengthening of condition (0).

Lemma 3.1.- *Let* $\mathcal{F}$ *be a game-family. Then for all* $X \in \mathcal{F}$ *and* $k \in \omega$*,* II *has no winning strategy in* $G_{\mathcal{F}}(k, X, \mathcal{F})$*.*

Proof – Assume that II has a winning strategy τ in $G_{\mathcal{F}}(k, X, \mathcal{F})$, where $X \in \mathcal{F}$ and $k > 0$. Let t denote the unique member of $(0, I_k)^0$. Note that if T belongs to $\mathcal{F}$, then so does $A(t, T)$ since $A(t, T)$ is obtained from T by amalgamation of blocks $T^{-1}(\{0\}), \ldots, T^{-1}(\{k\})$. We define a winning strategy ρ for II in $G_{\mathcal{F}}\big(0, A(t, X), \mathcal{F}\big)$ as follows. Let I's successive moves be

$Y_0, Y_1, \ldots$ Set $T_0 = D(t, X, Y_0)$ and $Z_0 = \tau(T_0)$. We put $\rho(Y_0) = A(t, Z_0)$. Then set $T_1 = D(t, Z_0, Y_1)$ and $Z_1 = \tau(T_0, T_1)$. We put $\rho(Y_0, Y_1) = A(t, Z_1)$, etc. We clearly have that $\prod_{i\in\omega} \rho(Y_0, \ldots, Y_i) = A\big(t, \prod_{i\in\omega} \tau(T_0, \ldots, T_i)\big)$. ■

Combinatorial applications of our games are mostly based on the following key lemma.

Lemma 3.2.- *Let* $\mathcal{F}$ *be a game-family, and let* σ *be a strategy for* I *in* $G_{\mathcal{F}}(k, X, \phi)$, *where* $k \in \omega$ *and* $X \in \mathcal{F}$. *Then* I *has a winning strategy* $\widehat{\sigma}$ *in* $G_{\mathcal{F}}(k, X, W)$, *where* W *consists of all* $Z \in (k, X)^\omega$ *such that for every* $T \in (k, Z)^\omega$, *there are* $R_0 \in \mathcal{F} \cap \big(k, \sigma(\phi)\big)^\omega$, $R_1 \in \mathcal{F} \cap \big(k+1, \sigma(R_0)\big)^\omega$, $R_2 \in \mathcal{F} \cap \big(k+2, \sigma(R_0, R_1)\big)^\omega, \ldots$ *with* $\prod_{i\in\omega} R_i = T$.

Proof – We define $\widehat{\sigma}$ as follows. Let II's successive moves be $Z_0, Z_1, \ldots$ We put $\widehat{\sigma}(\phi) = \sigma(\phi)$ and $\widehat{\sigma}(Z_0) = \sigma(Z_0)$.

Let t_r^0 for $r \leq k$ be an enumeration of the elements of $\big(k, I_{k+1}\big)^0$. We define S_r^0, Z_r^0, Y_r^0 and T_r^0 for $r \leq k$ as follows :

i) $S_0^0 = \sigma(Z_0, Z_1)$;

ii) $S_{j+1}^0 = T_j^0$;

iii) $Z_r^0 = A(t_r^0, S_r^0)$;

iv) $Y_r^0 = \sigma(Z_r^0)$;

v) $T_r^0 = D(t_r^0, S_r^0, Y_r^0)$.

We put $\widehat{\sigma}(Z_0, Z_1) = T_k^0$.

Let t_d^1 for $d \leq q$ be an enumeration of the elements of $(k, I_{k+2})^0 \cup (k, I_{k+2})^1$. We define S_d^1, Z_d^1, Y_d^1 and T_d^1 for $d \leq q$ as follows :

(0) $S_0^1 = \sigma(Z_0, Z_1, Z_2)$;

(1) $S_{j+1}^1 = T_j^1$;

(2) $Z_d^1 = A(t_d^1, S_d^1)$;

(3) If $t_d^1 \in (k, I_{k+2})^0$, $Y_d^1 = \sigma(Z_d^1)$;

(4) If $t_d^1 \in (k, I_{k+2})^1$ and $\ell(t_d^1, k+1) = k+1$, $Y_d^1 = \sigma(Z_0, Z_d^1)$;

(5) If $t_d^1 \in (k, I_{k+2})^1$ and $\ell(t_d^1, k+1) = k+2$, $Y_d^1 = \sigma(Z_r^0, Z_d^1)$, where r is such that $t_r^0 = t_d^1 \restriction (k+2)$;

(6) $T_d^1 = D(t_d^1, S_d^1, Y_d^1)$.

We put $\widehat{\sigma}(Z_0, Z_1, Z_2) = T_q^1$. etc.

Let $Z = \prod_{i\in\omega} Z_i$. Given $T \in (k, Z)^\omega$, we must find $R_0 \in \mathcal{F} \cap (k, \sigma(\phi))^\omega$, $R_1 \in \mathcal{F} \cap (k+1, \sigma(R_0))^\omega$, $R_2 \in \mathcal{F} \cap (k+2, \sigma(R_0, R_1))^\omega, \ldots$ such that $\prod_{i\in\omega} R_i = T$.

Let us first define R_0. If $\ell(T, k+1) = \ell(Z, k+1)$, set $R_0 = Z_0$. If $\ell(T, k+1) = \ell(Z, k+2)$, there is $u \in (k, I_{k+1})^0$ such that $T \in (k+1, A(u, Z))^\omega$. Then $u = t_r^0$ for some $r \leq k$. Set $R_0 = Z_r^0$. If $\ell(T, k+1) = \ell(Z, k+3)$, there is $v \in (k, I_{k+2})^0$ such that $T \in (k+1, A(v, Z))^\omega$. Let $v = t_d^1$, where $d \leq q$, and put $R_0 = Z_d^1$. etc.

Next, define R_1 as follows. If $\ell(T, k+2) = \ell(Z, k+2)$, set $R_1 = Z_1$. If $\ell(T, k+2) = \ell(Z, k+3)$, we must have $\ell(T, k+1) = \ell(Z, k+i)$, where $i = 1$ or $i = 2$. There is $w \in (k, I_{k+2})^1$ such that $\ell(w, k+1) = k+i$ and $T \in (k+2, A(w, Z))^w$. Then $w = t_e^1$ for some $e \leq q$. Set $R_1 = Z_e^1$. etc. ■

It follows that if $\mathcal{F}$ is a game-family and I has a winning strategy in $G_{\mathcal{F}}(k, X, K)$, where $k \in \omega$, $X \in \mathcal{F}$ and $K \subseteq (\omega)^\omega$, then I has a winning strategy in

$$G_{\mathcal{F}}(k, X, \{Z \in (k, X)^\omega : (k, Z)^\omega \subseteq K\}).$$

As a rule, Lemma 3.2 will be used in combination with the following, which easily follows from Lemma 3.1.

Lemma 3.3.- *Let* $\mathcal{F}$ *be a game-family. Then for all* $X \in \mathcal{F}$ *and* $k \in \omega$, I *has no winning strategy in* $G_{\mathcal{F}}(k, X, (\omega)^\omega - \mathcal{F})$.

Proof – If I had a winning strategy σ in $G_{\mathcal{F}}(k, X, (\omega)^\omega - \mathcal{F})$, then we could define a winning strategy τ for II in $G_{\mathcal{F}}(k, X, \mathcal{F})$ by letting $\tau(Y_0, \ldots, Y_i) = \sigma(Y_0, \ldots, Y_i)$, which would contradict Lemma 3.1. ■

It follows from Lemma 3.3 that if $\mathcal{F}$ is a game-family and I has a winning strategy in $G_{\mathcal{F}}(k, X, \mathcal{W})$, where $X \in \mathcal{F}$, $k \in \omega$ and $\mathcal{W} \subseteq \mathcal{F}^\omega$, then there is $(Z_0, Z_1, Z_2, \ldots) \in \mathcal{W}$ such that $\bigsqcap_{i \in \omega} Z_i \in \mathcal{F} \cap (k, X)^\omega$.

Let us now give some examples of game-families.

For each nonempty subset Q of $(\omega)^\omega$, we put

$$\mathcal{F}_Q = \{Y \in (\omega)^\omega : (\exists Z \in Q) Y \sqcap Z \in (\omega)^\omega\}.$$

Note that $\mathcal{F}_{(\omega)^\omega} = (\omega)^\omega$.

Proposition 3.4.- *Let* Q *be a nonempty subset of* $(\omega)^\omega$. *Then* $\mathcal{F}_Q$ *is a game-family.*

Proof – I has a clear winning strategy in $G_{\mathcal{F}_Q}(0, X, \mathcal{F}_Q)$ for all $X \in \mathcal{F}_Q$. ■

In order to introduce the next class of examples, we recall some definitions.

Let J be an ideal on ω, i.e. a proper subset of $P(\omega)$ such that (i) $P(A) \subseteq J$ for all $A \in J$, (ii) $A \cup B \in J$ for all $A, B \in J$, and (iii) $\{n\} \in J$ for all $n \in \omega$.

We put $J^+ = P(\omega) - J$.

J is *weakly selective* if given $A \in J^+$ and $f \in \omega^A$ with $\{f^{-1}(\{n\}) : n \in \omega\} \subseteq J$, there exists $C \in J^+ \cap P(A)$ such that f is one-to-one on C.

We set $[A]_J = \{B \subseteq \omega : (A - B) \cup (B - A) \in J\}$ for all $A \subseteq \omega$.

We let $[A]_J \leq [A']_J$ just in case $A - A' \in J$.

J is *σ-distributive* if the notion of forcing $({}^{P(\omega)}/_J - \{[\phi]_J\}, \leq_J)$ is σ-distributive, where ${}^{P(\omega)}/_J = \{[A]_J : A \subseteq \omega\}$.

Let us now set $\mathcal{F}_J = \{X \in (\omega)^\omega : \ell(X) \in J^+\}$, where $\ell(X) = \{\ell(X, i) : i \in \omega\}$.

Thus, $\mathcal{F}_J$ is the set of all $X \in (\omega)^\omega$ such that the set of leaders of X does not lie in J.

Proposition 3.5.- *Let J be an ideal on ω. Then $\mathcal{F}_J$ is a game-family if and only if J is σ-distributive and weakly selective.*

Proof – By Proposition 16.2 of [14], J is σ-distributive and weakly selective if and only if II has no winning strategy in the game $\mathcal{G}(C)$ for every $C \in J^+$, where $\mathcal{G}(C)$ is defined as follows : I picks $A_0 \in J^+ \cap P(C)$; then II picks $n_0 \in A_0$ and $B_0 \in J^+ \cap P\big(A_0 - (n_0 + 1)\big)$; then I picks $A_1 \in J^+ \cap P(B_0)$; then II picks $n_1 \in A_1$ and $B_1 \in J^+ \cap P\big(A_1 - (n_1 + 1)\big)$; etc. I is the winner if and only if $\{n_i : i \in \omega\} \in J^+$.

Let us now assume that there exists $X \in \mathcal{F}_J$ such that II has a winning strategy τ in $G_{\mathcal{F}_J}(0, X, \mathcal{F}_J)$. We define a winning strategy ρ for II in $\mathcal{G}\big(\ell(X)\big)$ as follows. Let I play $A_0, A_1, \ldots$. Select $Y_0 \in (X)^\omega$ with $\ell(Y_0) - \{0\} = A_0 - \{0\}$, and put

$$\rho(A_0) = \Big(\ell(\tau(Y_0), 1),\ \{\ell\big(\tau(Y_0), i\big) : i > 1\}\Big).$$

Select $Y_1 \in \big(1, \tau(Y_0)\big)^\omega$ with $\{\ell(Y_1, i) : i > 1\} = A_1$, and put

$$\rho(A_0, A_1) = \Big(\ell(\tau(Y_0, Y_1), 2), \{\ell\big(\tau(Y_0, Y_1), i\big) : i > 2\}\Big), \text{ etc.}$$

Conversely, assume that there is $C \in J^+$ such that II has a winning strategy σ in $\mathcal{G}(C)$. Pick $X \in (\omega)^\omega$ with $\ell(X) = C \cup \{0\}$. We define a winning strategy χ for II in $G_{\mathcal{F}_J}(0, X, \mathcal{F}_J)$ as follows. Let I's first move be Y_0. Let $(n_0, B_0) = \sigma\big(\ell(Y_0) - \{0\}\big)$, and pick $Z_0 \in (0, Y_0)^\omega$ with $\ell(Z_0) = \{0, n_0\} \cup B_0$. We put $\chi(Y_0) = Z_0$. Let I's next move be Y_1. Let $(n_1, B_1) = \sigma\big(\ell(Y_1) - \{0, n_0\}\big)$, and pick $Z_1 \in (1, Y_1)^\omega$ with $\ell(Z_1) = \{0, n_0, n_1\} \cup B_1$. We put $\chi(Y_0, Y_1) = Z_1$, etc. ∎

4. Generalizing Carlson's Lemma

The section is devoted to the proof of Lemma 4.1 below, which generalizes Carlson's Lemma (Lemma 2.4 in [2]). This last asserts that given $k \in \omega$, $X \in (\omega)^\omega$ and $\mathcal{D} \subseteq \, < X >_k$, there is $S \in (k, X)^\omega$ such that either $< S >_k \subseteq \mathcal{D}$ or else $< S >_k \cap \mathcal{D} = \phi$.

Throughout the section $\mathcal{F}$ will denote a fixed game-family.

LEMMA 4.1.- *Let $k \in \omega$, $X \in \mathcal{F}$ and $\mathcal{D} \subseteq \, < X >_k$. Then there exists $S \in \mathcal{F} \cap (k, X)^\omega$ such that either $< S >_k \subseteq \mathcal{D}$ or else $< S >_k \cap \mathcal{D} = \phi$.*

The proof of Lemma 4.1 is broken into several steps. We will use the Hales-Jewett Theorem [10] which reads as follows.

PROPOSITION 4.2.- *For all $n, c \in \omega - \{0\}$, there exists $h_{n,c} \in \omega - \{0\}$ with the following property : Given $F : n^{h_{n,c}} \to c$, there are $a \subset h_{n,c}, \varphi \in n^a$ and $m \in c$ such that $F(f) = m$ for all $f \in n^{h_{n,c}}$ such that $f \restriction a = \varphi$ and f is constant on $h_{n,c} - a$.*

Given $k \in \omega, X \in \mathcal{F}$ and $\mathcal{D} \subseteq < X >_k$, we put

$$\mathcal{W}_\mathcal{D} = \{(Z_0, Z_1, \ldots) \in \mathcal{F}^\omega : s(Z_0, k+1) \in \mathcal{D}\}$$
$$\mathcal{D}^+ = \{v \in < X >_{k+1} : \big(\forall t \in (k, I_{k+1})^0\big) \; A(t, v) \in \mathcal{D}\}$$
$$\mathcal{X}_\mathcal{D} = \{(Z_0, Z_1, \ldots) \in \mathcal{F}^\omega : s(Z_1, k+2) \in \mathcal{D}^+\}.$$

SUBLEMMA 4.3.- *Let $k \in \omega$, $X \in \mathcal{F}$ and $\mathcal{D} \subseteq < X >_k$ be such that* II *has a winning strategy in $G_\mathcal{F}(k, X, \mathcal{F}^\omega - \mathcal{W}_\mathcal{D})$. Then* II *has a winning strategy in $G_\mathcal{F}(k, X, \mathcal{F}^\omega - \mathcal{X}_\mathcal{D})$.*

Proof – Assume this is not the case. Then clearly I has a winning strategy in $G_\mathcal{F}(k, X, \mathcal{F}^\omega - \mathcal{X}_\mathcal{D})$. Hence by Lemmas 3.2 and 3.3, there is $X' \in \mathcal{F} \cap (k, X)^\omega$

with the property that given $Z \in \mathcal{F} \cap (k, X')^\omega$, there are $B_0^Z \in \mathcal{F} \cap \big(k, \sigma(\phi)\big)^\omega$, $B_1^Z \in \mathcal{F} \cap \big(k+1, \sigma(B_0^Z)\big)^\omega$, $B_2^Z \in \mathcal{F} \cap \big(k+2, \sigma(B_0^Z, B_1^Z)\big)^\omega, \ldots$ such that $\prod_{i \in \omega} B_i^Z = Z$. Then for any $Z \in \mathcal{F} \cap (k, X')^\omega$, $s(Z, k+2) \notin \mathcal{D}^+$ since $\big(B_0^Z, B_1^Z, \ldots\big) \notin \mathcal{X}_\mathcal{D}$ and $s(Z, k+2) = s(B_1^Z, k+2)$.

For each $q \in \omega$, set $e_q = q + 1 + h_{k+1,\ |(k+1)^{q+1}|}$, and let $\mathcal{Z}_q$ denote the set of all $Z \in (k, X')^\omega$ with the following property : Given $u \in (k, I_{k+e_q})^{q+1}$ such that $u^{-1}(\{k+1+j\}) = \{k+1+j\}$ for all $j \leq q$, there is $t \in (k, I_{k+1+q})^0$ such that $s\big(A(t, A(u, Z)), k+1\big) \in \mathcal{D}$. Let $\mathcal{T}$ be the set of all $(Z_0, Z_1, \ldots) \in \mathcal{F}^\omega$ such that $Z_{e_q} \in \mathcal{Z}_q$ for some $q \in \omega$.

We claim that II has a winning strategy in $G_\mathcal{F}(k, X', \mathcal{F}^\omega - \mathcal{T})$. Assume otherwise. Then by Lemma 2.1, I has a winning strategy σ in $G_\mathcal{F}(k, X', \mathcal{F}^\omega - \mathcal{T})$. Using Lemmas 3.2 and 3.3, we can find $S \in \mathcal{F} \cap (k, X')^\omega$ with the property that for every $T \in (k, S)^\omega$, there are $Z_0^T \in \mathcal{F} \cap (k, \sigma(\phi))^\omega$, $Z_1^T \in \mathcal{F} \cap \big(k+1, \sigma(Z_0^T)\big)^\omega$, $Z_2^T \in \mathcal{F} \cap \big(k+2, \sigma(Z_0^T, Z_1^T)\big)^\omega, \ldots$ such that $\prod_{i \in \omega} Z_i^T = T$. Set $W = \{R \in (k, S)^\omega : \mathcal{D} \cap < R >_k = \phi\}$. We define a winning strategy τ for II in $G_\mathcal{F}(k, S, (\omega)^\omega - W)$ as follows. Let I's successive moves be $T_0, T_1, \ldots$ We select $Q \in \mathcal{F} \cap (k, T_0)^\omega$ so that $s(Q, k+1) \notin \mathcal{D}$, and we put $\tau(T_0) = Q$. Let us now define $\tau(T_0, \ldots, T_{i+1})$. As $Z_{e_i}^{T_{i+1}} \notin \mathcal{Z}_i$, there exists $u \in (k, I_{k+e_i})^{i+1}$ such that a) $u^{-1}(\{k+1+j\}) = \{k+1+j\}$ for all $j \leq i$, and b) $s\big(A(t, A(u, Z_{e_i}^{T_{i+1}})), k+1\big) \notin \mathcal{D}$ for all $t \in (k, I_{k+1+q})^0$. We put $\tau(T_0, \ldots, T_{i+1}) = A(u, T_{i+1})$. We have by Lemma 3.1 that $\mathcal{F} \cap W \neq \phi$, which is contradictory.

By the claim there are $q \in \omega$ and $Z \in \mathcal{F} \cap (k, X')^\omega$ such that $Z \in \mathcal{Z}_q$. By the definition of $h_{k+1,\ |(k+1)^{q+1}|}$, there exist $t \in (k, I_{k+1+q})^0$ and $v \in (k, I_{k+e_q-(q+1)})^1$ such that $s(A(v, A(t, Z)), k+2) \in \mathcal{D}^+$, which yields the desired contradiction.

∎

SUBLEMMA 4.4.- *Let* $k \in \omega$, $X \in \mathcal{F}$ *and* $\mathcal{D} \subseteq < X >_k$ *be such that* II *has a winning strategy in* $G_\mathcal{F}(k, X, \mathcal{F}^\omega - \mathcal{W}_\mathcal{D})$. *Then* II *has a winning strategy in* $G_\mathcal{F}\big(k, X, \mathcal{F}^\omega - (\mathcal{W}_\mathcal{D} \cap \mathcal{X}_\mathcal{D})\big)$.

Proof – Define $\mathcal{D}_i$ for $i \in \omega$ by letting $\mathcal{D}_0 = \mathcal{D}^+$ and $\mathcal{D}_{i+1} = \mathcal{D}_i^+$. We define a winning strategy τ for II in $G_\mathcal{F}(k, X, \mathcal{F}^\omega - (\mathcal{W}_\mathcal{D} \cap \mathcal{X}_\mathcal{D})\big)$ as follows. Given $Y \in \mathcal{F} \cap (k, X)^\omega$, use Sublemma 4.3 to define a strategy σ for II in $G_\mathcal{F}(k, Y, \phi)$ so that for all i, II has a winning strategy in $G_\mathcal{F}(k+1+i,\ \sigma(Y_0, \ldots, Y_i), \mathcal{F}^\omega - \mathcal{W}_{\mathcal{D}_i})$. By Lemma 3.1 there are $Z \in \mathcal{F}$ and $Y_0^Z \in \mathcal{F} \cap (k, Y)^\omega, Y_1^Z \in \mathcal{F} \cap \big(k+1, \sigma(Y_0^Z)\big)^\omega, Y_2^Z \in \mathcal{F} \cap \big(k+2, \sigma(Y_0^Z, Y_1^Z)\big)^\omega, \ldots$ such that $Z = \prod_{i \in \omega} \sigma(Y_0^Z, \ldots, Y_i^Z)$. Pick $i \in \omega$ and $t \in (k, I_{k+1+i})^1$

so that $\ell(t, k+1) = k+1+i$ and $s(A(t,Z), k+1) \in \mathcal{D}$. We put $\tau(Y) = A\big(t, \sigma(Y_0^Z, \ldots, Y_i^Z)\big)$. By the definition of σ, II has a winning strategy ρ in $G_{\mathcal{F}}(k+1+i, \sigma(Y_0^Z, \ldots, Y_i^Z), \mathcal{F}^\omega - \mathcal{W}_{\mathcal{D}_i})$. We put

$$\tau(Y, Y') = A\big(t, \rho\big(D(t, \sigma(Y_0^Z, \ldots, Y_i^Z), Y')\big)\big)$$

for all $Y' \in \mathcal{F} \cap \big(k+1, \tau(Y)\big)^\omega$. ■

Proof of Lemma 4.1. – The game $G_{\mathcal{F}}(k, X, \mathcal{F}^\omega - \mathcal{W}_{\mathcal{D}})$ is clearly determined. If I has a winning strategy in the game, then there exists $T \in \mathcal{F} \cap (k, X)^\omega$ with the property that $\mathcal{D} \cap < T >_k = \phi$. Now suppose II has a winning strategy in the game. Let W be the set of all $S \in (k, X)^\omega$ such that $< S >_k \subseteq \mathcal{D}$. Define $\mathcal{E}_i$ for $i \in \omega$ by letting $\mathcal{E}_0 = \mathcal{D}$ and $\mathcal{E}_{i+1} = \mathcal{E}_i^+$. Use Sublemma 4.4 to define a strategy τ for II in $G_{\mathcal{F}}\big(k, X, (\omega)^\omega - W\big)$ so that for all i, $s\big(\tau(Y_0, \ldots, Y_i), k+1+i\big) \in \mathcal{E}_i$ and II has a winning strategy in $G_{\mathcal{F}}(k+1+i, \tau(Y_0, \ldots, Y_i), \mathcal{F}^\omega - \mathcal{W}_{\mathcal{E}_{i+1}})$. As the strategy τ is clearly a winning one, we have by Lemma 3.1 that $W \cap \mathcal{F} \neq \phi$. ■

5. $P_{\mathcal{F}}$

Throughout the section $\mathcal{F}$ will denote a fixed game-family.

We set $P_{\mathcal{F}} = \omega \times \mathcal{F}$ and order $P_{\mathcal{F}}$ by letting $(k', X') \leq (k, X)$ just in case $(k', X')^\omega \subseteq (k, X)^\omega$.

Notice that $(k', X') \leq (k, X)$ if and only if $k' \geq k$ and $X' \in (k, X)^\omega$.

Just as any other poset, $(P_{\mathcal{F}}, \leq)$ can be viewed as a notion of forcing, which, in case $\mathcal{F} = (\omega)^\omega$, is known as dual Mathias forcing (see [2]).

In this section we describe some key properties of $(P_{\mathcal{F}}, \leq)$. The treatment of this material uses ideas of [15] and [16].

$\Lambda \subseteq P_{\mathcal{F}}$ is *inductive* if we have that $(k, X) \in \Lambda$ if and only if $(k+1, Y) \in \Lambda$ whenever $(k+1, Y) \in P_{\mathcal{F}}$ is such that $(k+1, Y) \leq (k, X)$.

We observe that if $\Lambda \subseteq P_{\mathcal{F}}$ is inductive, then it is open, which means that $\{(k', X') \in P_{\mathcal{F}} : (k', X') \leq (k, X)\} \subseteq \Lambda$ for every $(k, X) \in \Lambda$.

The following collects some easy examples of inductive sets.

Lemma 5.1.-

(i) $\{(k,X)\in P_{\mathcal{F}}:(k,X)^{\omega}\subseteq W\}$ *is inductive for every* $W\subseteq(\omega)^{\omega}$.

(ii) $\{(k,X)\in P_{\mathcal{F}}:\mathcal{F}\cap(k,X)^{\omega}\subseteq W\}$ *is inductive for every* $W\subseteq\mathcal{F}$.

(iii) *Let* $n\in\omega-\{0\}$. *Then* $\{(k,X)\in P_{\mathcal{F}}:\big(k\cap(n-1),X\big)^{(n-1)-(k\cap(n-1))}\subseteq W\}$ *is inductive for every* $W\subseteq(\omega)^{n}$.

(iv) $\{(k,X)\in P_{\mathcal{F}}:(k,X)\Vdash\varphi\}$ *is inductive for every sentence* φ *of the forcing language of* $(P_{\mathcal{F}},\leq)$.

We will see that the notion of forcing $(\mathcal{P}_{\mathcal{F}},\leq)$ has pure decision.

Lemma 5.2.- *Let* $(k,X)\in P_{\mathcal{F}}$ *and an inductive* $\Lambda\subseteq P_{\mathcal{F}}$ *be such that* $(k,Z)\notin\Lambda$ *for all* $Z\in\mathcal{F}\cap(k,X)^{\omega}$. *Then there exists* $Z\in\mathcal{F}\cap(k,X)^{\omega}$ *with the property that* $(k',Z')\notin\Lambda$ *for all* $(k',Z')\in P_{\mathcal{F}}$ *with* $(k',Z')\leq(k,Z)$.

Proof –
Claim 1. There exists $X'\in\mathcal{F}\cap(k,X)^{\omega}$ with the following property : Let $t\in(k,I_{k+i})^{j}$, where $j\leq i<\omega$, be such that $\ell(t,k+j)=k+i$. If $(k+j,T)\in\Lambda$ for some $T\in\mathcal{F}\cap\big(k+j,A(t,X')\big)^{\omega}$, then $\big(k+j,A(t,X')\big)\in\Lambda$.

Proof of Claim 1. We start by defining a strategy σ for I in $G_{\mathcal{F}}(k,X,\phi)$ as follows. We put $\sigma(\phi)=X$. Let us now define $\sigma(Z_0,\dots,Z_i)$. Put

$$\mathcal{Z}=\{Y\in\mathcal{F}\cap(k+1+i,Z_i)^{\omega}:(k+1+i,Y)\in\Lambda\}.$$

If $\mathcal{Z}\neq\phi$, then pick $Y\in\mathcal{Z}$ and put $\sigma(Z_0,\dots,Z_i)=Y$. Otherwise put $\sigma(Z_0,\dots,Z_i)=Z_i$.

Now let $\widehat{\sigma}$ be defined as in the proof of Lemma 3.2. By Lemma 3.3, there are $S_0\in\mathcal{F}\cap(k,\widehat{\sigma}(\phi))^{\omega}$, $S_1\in\mathcal{F}\cap\big(k+1,\widehat{\sigma}(S_0)\big)^{\omega}$, $S_2\in\mathcal{F}\cap\big(k+2,\widehat{\sigma}(S_0,S_1)\big)^{\omega},\dots$ such that $\bigsqcap_{i\in\omega}S_i\in\mathcal{F}$. We put $X'=\bigsqcap_{i\in\omega}S_i$.

Let i,j with $0<j\leq i<\omega$, $t\in(k,I_{k+i})^{j}$ with $\ell(t,k+j)=k+i$, and $T\in\mathcal{F}\cap(k+j,A(t,X'))^{\omega}$ with $(k+j,T)\in\Lambda$. Set $u=t\restriction k+i$. There exist $Z_0^T\in\mathcal{F}\cap(k,\sigma(\phi))^{\omega}$, $Z_1^T\in\mathcal{F}\cap\big(k+1,\sigma(Z_0^T)\big)^{\omega}$, $Z_2^T\in\mathcal{F}\cap\big(k+2,\sigma(Z_0^T,Z_1^T)\big)^{\omega},\dots$ such that $\bigsqcap_{r\in\omega}Z_r^T=T$. As $T\in\mathcal{F}\cap\big(k+j,Z_{j-1}^T\big)^{\omega}$, we have that $\big(k+j,\sigma(Z_0^T,\dots,Z_{j-1}^T)\big)\in\Lambda$. It follows that $\big(k+j,\ A(u,\widehat{\sigma}(S_0,\dots,S_i))\big)\in\Lambda$, since $A\big(u,\widehat{\sigma}(S_0,\dots,S_i)\big)\in\big(k+j,\sigma(Z_0^T,\dots,Z_{j-1}^T)\big)^{\omega}$. As $X'\in\big(k+i,\widehat{\sigma}(S_0,\dots,S_i)\big)^{\omega}$ and $A\big(u,\widehat{\sigma}(S_0,\dots,S_i)\big)=A\big(t,\widehat{\sigma}(S_0,\dots,S_i)\big)$, we have that $(k+j,A(t,X'))\in\Lambda$, which concludes the proof of Claim 1.

Claim 2. I has a winning strategy ρ in $G_{\mathcal{F}}(k, X', \mathcal{W})$, where $\mathcal{W}$ consists of all $(Z_0, Z_1, \ldots) \in \mathcal{F}^\omega$ such that $(k+1+i, T) \notin \Lambda$ for all $i \in \omega$ and $T \in \mathcal{F} \cap (k+1+i, Z_i)^\omega$.

Proof of Claim 2. Define

$$\varphi :< X' >_k \to \bigcup_{i \geq 1} \{t \in (k, I_{k+i})^1 : \ell(t, k+1) = k+i\}$$

so that for all $u \in < X' >_k$, $u = s(A(\varphi(u), X'), k+1)$. Then define $F :< X' >_k \to 2$ by letting $F(u) = 0$ if and only if $(k+1, A(\varphi(u), X')) \in \Lambda$. By Lemma 4.1, there are $Y_0 \in \mathcal{F} \cap (k, X')^\omega$ and $e_0 \in 2$ such that F takes the constant value e_0 on $< Y_0 >_k$. We have that $e_0 = 1$, since $Y_0 \in \mathcal{F} \cap (k, X)^\omega$ and for every $(k+1, T) \in P_{\mathcal{F}}$ with $(k+1, T) \leq (k, Y_0)$, there is $u \in < Y_0 >_k$ such that $(k+1, T) \leq (k+1, A(\varphi(u), X'))$. If $T \in \mathcal{F} \cap (k, Y_0)^\omega$ then setting $u = s(T, k+1)$, we have that $F(u) = 1$ and $T \in \mathcal{F} \cap \big(k+1, A(\varphi(u), X')\big)^\omega$, and therefore $(k+1, T) \notin \Lambda$. We put $\rho(\phi) = Y_0$.

Now let II play Z_0. Clearly $(k+1, T) \notin \Lambda$ for all $T \in \mathcal{F} \cap (k+1, Z_0)^\omega$. Define

$$\psi : < Z_0 >_{k+1} \to \bigcup_{i \geq 1} \{t \in (k+1, I_{k+1+i})^1 : \ell(t, k+2) = k+1+i\}$$

so that for all $v \in < Z_0 >_{k+1}$, $v = s\big(A(\psi(v), Z_0), k+2\big)$. Then define $H : < Z_0 >_{k+1} \to 2$ by letting $H(v) = 0$ if and only if $\big(k+2, A(\psi(v), Z_0)\big) \in \Lambda$. By Lemma 4.1, there exists $Y_1 \in \mathcal{F} \cap (k+1, Z_0)^\omega$ such that H is constant on $< Y_1 >_{k+1}$. It is readily checked that $(k+2, T) \notin \Lambda$ for all $T \in \mathcal{F} \cap (k+1, Y_1)^\omega$. We put $\rho(Z_0) = Y_1$, etc. This completes the proof of Claim 2.

By Lemmas 3.2 and 3.3, there exists $Z \in \mathcal{F} \cap (k, X)^\omega$ such that for every $T \in \mathcal{F} \cap (k, Z)^\omega$, there are $Z_0 \in \mathcal{F} \cap (k, \rho(\phi))^\omega$, $Z_1 \in \mathcal{F} \cap \big(k+1, \rho(Z_0)\big)^\omega$, $Z_2 \in \mathcal{F} \cap \big(k+2, \rho(Z_0, Z_1)\big)^\omega, \ldots$ with $\bigsqcap_{i \in \omega} Z_i = T$. Then $(k+1+i, T) \notin \Lambda$ for all $T \in \mathcal{F} \cap (k, Z)^\omega$ and $i \in \omega$. ∎

It follows from Lemma 5.2 that if $\Lambda \subseteq P_{\mathcal{F}}$ is inductive, then Λ is dense in $P_{\mathcal{F}}$ if and only if for every $(k, X) \in P_{\mathcal{F}}$, there is $Z \in \mathcal{F} \cap (k, X)^\omega$ such that $(k, Z) \in \Lambda$.

PROPOSITION 5.3.- *Let φ be a sentence of the forcing language of $(P_{\mathcal{F}}, \leq)$. Then for every $(k, X) \in P_{\mathcal{F}}$, there exists $Z \in \mathcal{F} \cap (k, X)^\omega$ such that either $(k, Z) \Vdash \varphi$ or else $(k, Z) \Vdash \neg\, \varphi$.*

Proof – By Lemmas 5.1. (iv) and 5.2. ∎

Proposition 5.3 generalizes Lemma 5.2 of [2].

PROPOSITION 5.4.- *Let* Λ_0, Λ_1 *be inductive subsets of* $P_{\mathcal{F}}$. *Then* $\Lambda_0 \cup \Lambda_1$ *is dense in* $P_{\mathcal{F}}$ *if and only if for every* $(k, X) \in P_{\mathcal{F}}$, *there exists* $Z \in \mathcal{F} \cap (k, X)^{\omega}$ *with* $(k, Z) \in \Lambda_0 \cup \Lambda_1$.

Proof – The right-to-left implication is obvious. To show the other implication, assume that $(k, X) \in P_{\mathcal{F}}$ is such that $(k, Z) \notin \Lambda_0 \cup \Lambda_1$ for all $Z \in \mathcal{F} \cap (k, X)^{\omega}$. By Lemma 5.2, there exists $Z_0 \in \mathcal{F} \cap (k, X)^{\omega}$ such that $(q, Y) \notin \Lambda_0$ for all $(q, Y) \in P_{\mathcal{F}}$ with $(q, Y) \leq (k, Z_0)$. Again by Lemma 5.2, there exists $Z_1 \in \mathcal{F} \cap (k, Z_0)^{\omega}$ such that $(r, T) \notin \Lambda_1$ for all $(r, T) \in P_{\mathcal{F}}$ with $(r, T) \leq (k, Z_1)$. Thus $\Lambda_0 \cup \Lambda_1$ is not dense in $P_{\mathcal{F}}$, since $(r, T) \notin \Lambda_0 \cup \Lambda_1$ for all $(r, T) \in P_{\mathcal{F}}$ with $(r, T) \leq (k, Z_1)$. ∎

Finally, let us show that the collection of all inductive dense subsets of $P_{\mathcal{F}}$ is closed under countable intersections.

LEMMA 5.5.- *Let* Λ_i *be an inductive subset of* $P_{\mathcal{F}}$ *for each* $i \in \omega$, *and let* $(k, X) \in P_{\mathcal{F}}$ *be such that* $(k, Z) \notin \bigcap_{i \in \omega} \Lambda_i$ *for every* $Z \in \mathcal{F} \cap (k, X)^{\omega}$. *Then* II *has a winning strategy in* $G_{\mathcal{F}}(k, X, \mathcal{F}^{\omega} - \mathcal{W})$, *where* $\mathcal{W}$ *consists of all* $(Z_0, Z_1, \ldots) \in \mathcal{F}^{\omega}$ *with the property that there exists* $i \in \omega$ *such that* $(k+1+i, S) \notin \Lambda_i$ *for all* $S \in \mathcal{F} \cap (k+1+i, Z_i)^{\omega}$.

Proof – Assume otherwise. Then by Lemma 2.1, I has a winning strategy ρ in $G_{\mathcal{F}}(k, X, \mathcal{F}^{\omega} - \mathcal{W})$.

Let us define a strategy σ for I in $G_{\mathcal{F}}(k, X, \phi)$ as follows. We put $\sigma(\phi) = \rho(\phi)$. Given $Z_0 \in \mathcal{F} \cap (k, \rho(\phi))^{\omega}$, we select $S_0 \in \mathcal{F} \cap (k+1, Z_0)^{\omega}$ so that $(k+1, S_0)^{\omega} \in \Lambda_0$, and we put $\sigma(Z_0) = \rho(S_0)$. Given $Z_1 \in \mathcal{F} \cap \left(k+1, \sigma(Z_0)\right)^{\omega}$, we select $S_1 \in \mathcal{F} \cap (k+2, Z_1)^{\omega}$ so that $(k+2, S_1) \in \Lambda_1$, and we put $\sigma(Z_0, Z_1) = \rho(S_0, S_1)$, etc.

By Lemmas 3.2 and 3.3, we can find $Z \in \mathcal{F} \cap (k, X)^{\omega}$ such that for every $T \in \mathcal{F} \cap (k, Z)^{\omega}$, there are $Z_0^T \in \mathcal{F} \cap (k, \sigma(\phi))^{\omega}$, $Z_1^T \in \mathcal{F} \cap \left(k+1, \sigma(Z_0^T)\right)^{\omega}$, $Z_2^T \in \mathcal{F} \cap \left(k+2, \sigma(Z_0^T, Z_1^T)\right)^{\omega}, \ldots$ with $\bigsqcap_{i \in \omega} Z_i^T = T$.

For each $T \in \mathcal{F} \cap (k, Z)^{\omega}$, we have that $(k+1, T) \in \Lambda_0$, as $(k+1, T) \leq (k+1, \sigma(Z_0^T))$ and $(k+1, \sigma(Z_0^T)) \in \Lambda_0$. Hence $(k, Z) \in \Lambda_0$.

Let $Y \in \mathcal{F} \cap (k, Z)^{\omega}$. For each $T \in \mathcal{F} \cap (k+1, Y)^{\omega}$, we have that $(k+2, T) \in \Lambda_1$, as $(k+2, T) \leq (k+2, \sigma(Z_0^T, Z_1^T))$ and $(k+2, \sigma(Z_0^T, Z_1^T)) \in \Lambda_1$. Hence $(k+1, Y) \in \Lambda_1$. It follows that $(k, Z) \in \Lambda_1$, etc.

Thus $(k, Z) \in \bigcap_{i \in \omega} \Lambda_i$, which brings the desired contradiction. ∎

PROPOSITION 5.6.- *Let* $\Lambda_i \subseteq P_{\mathcal{F}}$ *be inductive and dense in* $P_{\mathcal{F}}$ *for each* $i \in \omega$. *Then* $\bigcap_{i\in\omega} \Lambda_i$ *is inductive and dense in* $P_{\mathcal{F}}$.

Proof – This is immediate from Proposition 5.4 and Lemma 5.5. ∎

6. Generalizing the Dual Ellentuck Theorem 1 : the Ellentuck way

The main result of this section is Theorem 6.4, which generalizes the Dual Ellentuck Theorem of Carlson and Simpson [2]. It is stated in topological terms, just as Ellentuck's original theorem and its dual version.

Throughout this section $\mathcal{F}$ will denote a fixed game-family.

We let $\mathcal{N}_{\mathcal{F}}$ (respectively, $\mathcal{C}_{\mathcal{F}}$) consist of all $W \subseteq \mathcal{F}$ with the property that for all $(k, X) \in P_{\mathcal{F}}$, there exists $Z \in \mathcal{F} \cap (k, X)^{\omega}$ such that $W \cap (k, Z)^{\omega} = \phi$ (respectively, such that either $\mathcal{F} \cap (k, Z)^{\omega} \subseteq W$ or $W \cap (k, Z)^{\omega} = \phi$).

Note that we only consider subsets W of $\mathcal{F}$. For another approach dealing with all subsets of $(\omega)^{\omega}$, see the next section. The $\mathcal{N}$ in $\mathcal{N}_{\mathcal{F}}$ stands for "dual Ramsey null" and the $\mathcal{C}$ in $\mathcal{C}_{\mathcal{F}}$ for "completely dual Ramsey".

We let $\overline{\mathcal{N}}_{\mathcal{F}}$ (respectively, $\overline{\mathcal{C}}_{\mathcal{F}}$) consist of all $W \subseteq \mathcal{F}$ with the property that the set of all $(k, X) \in P_{\mathcal{F}}$ such that $W \cap (k, X)^{\omega} = \phi$ (respectively, such that either $\mathcal{F} \cap (k, X)^{\omega} \subseteq W$ or $W \cap (k, X)^{\omega} = \phi$) is dense in $P_{\mathcal{F}}$.

LEMMA 6.1.- *The following hold :*

(i) $\mathcal{C}_{\mathcal{F}} = \overline{\mathcal{C}}_{\mathcal{F}}$.
(ii) $\mathcal{N}_{\mathcal{F}} = \overline{\mathcal{N}}_{\mathcal{F}}$.
(iii) $\overline{\mathcal{N}}_{\mathcal{F}}$ *is closed under countable unions.*

Proof – (i) and (ii) : By Lemma 5.1 (ii) and Proposition 5.4.
(iii) : By Lemma 5.1 (ii) and Proposition 5.6. ∎

Let $\mathcal{X}$ be a topological space, and let $\mathcal{P}_{\mathcal{X}}$ denote the collection of all its nonempty basic open sets.

We let $\overline{\mathcal{N}}(\mathcal{X})$ (respectively, $\overline{\mathcal{C}}(\mathcal{X})$) consist of all $W \subseteq \mathcal{X}$ such that $\{p \in \mathcal{P}_{\mathcal{X}} : p \cap W = \phi\}$ (respectively, $\{p \in \mathcal{P}_{\mathcal{X}} : p \subseteq W$ or $p \cap W = \phi\}$) is dense in $(\mathcal{P}_{\mathcal{X}}, \subseteq)$.

LEMMA 6.2.- *Given $W \subseteq \mathcal{X}$, we have the following :*

(i) $W \in \overline{\mathcal{N}}(\mathcal{X})$ *if and only if* W *is nowhere dense.*
(ii) $W \in \overline{\mathcal{C}}(\mathcal{X})$ *if and only if there is an open set* U *such that* $W \Delta U$ *is nowhere dense.*

Proof – Set $O_S = \{p \in \mathcal{P}_{\mathcal{X}} : p \subseteq S\}$ for every $S \subseteq \mathcal{X}$. Then $W \in \overline{\mathcal{N}}(\mathcal{X})$ if and only if $O_{\mathcal{X}-W}$ is dense if and only if W is nowhere dense, which proves (i).

To prove the left-to-right implication of (ii), it suffices to observe that if $W \in \overline{\mathcal{C}}(\mathcal{X})$, then $O_W \cup O_{\mathcal{X}-W}$ is dense and $(W - O_W) \cap (O_W \cup O_{\mathcal{X}-W}) = \phi$. The reverse implication easily follows from (i) and the fact that $U \in \overline{\mathcal{C}}(\mathcal{X})$ for every open set U.

■

LEMMA 6.3.- *Assume that $\overline{\mathcal{N}}(\mathcal{X})$ is closed under countable unions. Then $\overline{\mathcal{C}}(\mathcal{X})$ is the set of all $W \subseteq \mathcal{X}$ which have the Baire property.*

Proof – Immediate from Lemma 6.2.

■

Now assume that $\{Z \in (\omega)^\omega : Y \leq Z\} \subseteq \mathcal{F}$ for all $Y \in \mathcal{F}$. If (k, X), $(k', X') \in P_{\mathcal{F}}$ are such that $\big(\mathcal{F} \cap (k, X)^\omega\big) \cap \big(\mathcal{F} \cap (k', X')^\omega\big) \neq \phi$, then it is readily checked that

$$\big(\mathcal{F} \cap (k, X)^\omega\big) \cap \big(\mathcal{F} \cap (k', X')^\omega\big) = \mathcal{F} \cap (k \cup k',\ X \sqcap X')^\omega.$$

Hence we can put a topology on $\mathcal{F}$ by taking as basic open sets ϕ and all sets of the form $\mathcal{F} \cap (k, X)^\omega$, where $(k, X) \in P_{\mathcal{F}}$. It is simple to see that $\overline{\mathcal{N}}(\mathcal{F}) = \overline{\mathcal{N}}_{\mathcal{F}}$ and $\overline{\mathcal{C}}(\mathcal{F}) = \overline{\mathcal{C}}_{\mathcal{F}}$.

THEOREM 6.4.- *Assume that $\mathcal{F}$ is closed under refinement. Then $\mathcal{C}_{\mathcal{F}}$ is the set of all $W \subseteq \mathcal{F}$ which have the Baire property.*

Proof – By Lemmas 6.1 and 6.3.

■

The Dual Ellentuck Theorem (Theorem 4.1 in [2]) asserts that $\mathcal{C}_{(\omega)^\omega}$ is the set of all $W \subseteq (\omega)^\omega$ which have the Baire property. It is an immediate consequence of Theorem 6.4 since $(\omega)^\omega$ is obviously closed under refinement.

7. Generalizing the Dual Ellentuck Theorem 2 : the Mathias way

The purpose of this section is again to generalize the Dual Ellentuck Theorem. We will this time follow Mathias, whose version of Ellentuck's Theorem (see Proposition 4.12 in [17]) is stated in terms of dense subsets of partial orderings, and not in topological terms. Let us remark that already in the previous section, the Ellentuck-type Theorem 6.4 was derived from the Mathias-type Lemma 6.1.

Part of the appeal of Ellentuck's Theorem lies in its compactness. Here is one statement which serves several purposes, e.g. 1) it gives a characterization of the completely Ramsey sets, 2) it shows that every Ellentuck neighborhood is completely Ramsey, and 3) it shows that completely Ramsey sets are closed under the operation $\mathcal{A}$. When one generalizes Ellentuck's Theorem in a context where a topological formulation is not handy, as Mathias did, tasks 1), 2), 3) have to be handled separately. We are faced with a situation of this type in the present section, and so our generalization of the Dual Ellentuck theorem is made of the combination of three theorems, namely Theorem 7.1, Theorem 7.6 and Theorem 7.7. We also have to check that the Dual Ellentuck Theorem can be deduced from our so-called generalization of it. This we do at the end of the section.

Throughout this section $\mathcal{F}$ will denote a fixed game-family.

We let $\mathcal{N}^{\omega}_{\mathcal{F}}$ (respectively, $\mathcal{C}^{\omega}_{\mathcal{F}}$) consist of all $W \subseteq (\omega)^{\omega}$ with the property that for every $(k, X) \in P_{\mathcal{F}}$, there exists $Z \in \mathcal{F} \cap (k, X)^{\omega}$ such that $W \cap (k, Z)^{\omega} = \phi$ (respectively, such that either $(k, Z)^{\omega} \subseteq W$ or $W \cap (k, Z)^{\omega} = \phi$).

We let $\overline{\mathcal{N}}^{\omega}_{\mathcal{F}}$ (respectively, $\overline{\mathcal{C}}^{\omega}_{\mathcal{F}}$) consist of all $W \subseteq (\omega)^{\omega}$ with the property that the set of all $(k, X) \in P_{\mathcal{F}}$ such that $W \cap (k, X)^{\omega} = \phi$ (respectively, such that either $(k, X)^{\omega} \subseteq W$ or $W \cap (k, X)^{\omega} = \phi$) is dense in $P_{\mathcal{F}}$.

We observe that $\mathcal{N}_{\mathcal{F}} = \mathcal{N}^{\omega}_{\mathcal{F}} \cap P(\mathcal{F})$.

Theorem 7.1.- *The following hold :*

(i) $\mathcal{C}^{\omega}_{\mathcal{F}} = \overline{\mathcal{C}}^{\omega}_{\mathcal{F}}$.

(ii) $\mathcal{N}^{\omega}_{\mathcal{F}} = \overline{\mathcal{N}}^{\omega}_{\mathcal{F}}$.

(iii) $\overline{\mathcal{N}}^{\omega}_{\mathcal{F}}$ *is closed under countable unions.*

Proof – (i) and (ii) : By Lemma 5.1 (i) and Proposition 5.4.
(iii) : By Lemma 5.1 (i) and Proposition 5.6. ∎

When $\mathcal{F} = (\omega)^{\omega}$, Theorem 7.1 is equivalent to Lemma 6.1 since $\mathcal{N}^{\omega}_{(\omega)^{\omega}}$, $\mathcal{C}^{\omega}_{(\omega)^{\omega}}$, $\overline{\mathcal{N}}^{\omega}_{(\omega)^{\omega}}$ and $\overline{\mathcal{C}}^{\omega}_{(\omega)^{\omega}}$ are respectively equal to $\mathcal{N}_{(\omega)^{\omega}}$, $\mathcal{C}_{(\omega)^{\omega}}$, $\overline{\mathcal{N}}_{(\omega)^{\omega}}$ and $\overline{\mathcal{C}}_{(\omega)^{\omega}}$.

$\mathcal{C}_{\mathcal{F}}^{\omega}$ and $\mathcal{N}_{\mathcal{F}}^{\omega}$ can also be described in the language of games. We will use the following fact.

LEMMA 7.2.- *Let* $(k, X) \in P_{\mathcal{F}}$ *and* $W \subseteq (\omega)^{\omega}$ *be such that* II *has a winning strategy in* $G_{\mathcal{F}}(k, X, W)$. *Then* I *has a winning strategy in* $G_{\mathcal{F}}(k, T, (\omega)^{\omega} - W)$ *for all* $T \in \mathcal{F} \cap (k, X)^{\omega}$.

Proof – Fix $T \in \mathcal{F} \cap (k, X)^{\omega}$, and select a winning strategy τ for II in $G_{\mathcal{F}}(k, X, W)$. We define a winning strategy σ for I in $G_{\mathcal{F}}(k, T, (\omega)^{\omega} - W)$ as follows. Put $R_0 = \bigcup_{Y \in \mathcal{F} \cap (k,T)^{\omega}} (k+1, \tau(Y))^{\omega}$. By Lemmas 5.1 (i) and 5.2, there exists $S_0 \in \mathcal{F} \cap (k, T)^{\omega}$ such that $(k, S_0)^{\omega} \subseteq R_0$. We set $\sigma(\phi) = S_0$. Let II's answer to S_0 be Z_0. Pick $Y_0 \in \mathcal{F} \cap (k, T)^{\omega}$ so that $Z_0 \in \big(k+1, \tau(Y_0)\big)^{\omega}$. Now put $R_1 = \bigcup_{Y \in \mathcal{F} \cap (k+1, Z_0)^{\omega}} \big(k+2, \tau(Y_0, Y)\big)^{\omega}$. By Lemmas 5.1 (i) and 5.2, there is $S_1 \in \mathcal{F} \cap (k+1, Z_0)^{\omega}$ with $(k+1, S_1)^{\omega} \subseteq R_1$. We set $\sigma(Z_0) = S_1$, etc. ∎

PROPOSITION 7.3.- *Given* $W \subseteq (\omega)^{\omega}$, *the following are equivalent :*

(i) $W \in \mathcal{N}_{\mathcal{F}}^{\omega}$.
(ii) II *has a winning strategy in* $G_{\mathcal{F}}(k, X, W)$ *for every* $(k, X) \in P_{\mathcal{F}}$.
(iii) I *has a winning strategy in* $G_{\mathcal{F}}(k, X, (\omega)^{\omega} - W)$ *for every* $(k, X) \in P_{\mathcal{F}}$.

Proof – (i) $\rightarrow$ (ii) : Clear.
(ii) $\rightarrow$ (iii) : By Lemma 7.2.
(iii) $\rightarrow$ (i) : By Lemmas 3.2 and 3.3. ∎

PROPOSITION 7.4.- *Given* $W \subseteq (\omega)^{\omega}$, *the following are equivalent :*
(i) $W \in \mathcal{C}_{\mathcal{F}}^{\omega}$.
(ii) *The game* $G_{\mathcal{F}}(k, X, W)$ *is determined for every* $(k, X) \in P_{\mathcal{F}}$.

Proof – (i) $\rightarrow$ (ii) : Clear.
(ii) $\rightarrow$ (i) : By Lemmas 7.2, 3.2 and 3.3. ∎

We mention that in particular, Propositions 7.3 and 7.4 provide new characterizations of $\mathcal{N}_{(\omega)^{\omega}}$ and $\mathcal{C}_{(\omega)^{\omega}}$.

LEMMA 7.5.- *Given* $(k,X) \in P_{(\omega)^\omega}$ *and* $(k',X') \in P_{\mathcal{F}}$, *there exists* $(k'',X'') \in P_{\mathcal{F}}$ *such that* $k'' \leq k \cup k' \cup 1, (k'',X'') \leq (k',X')$ *and either* $(k'',X'')^\omega \subseteq (k,X)^\omega$ *or* $(k'',X'')^0 \cap (k,X))^{k''-k} = \phi$.

Proof – If $X' \in (k,X)^\omega$, then $(k \cup k', X')^\omega \subseteq (k,X)^\omega$. If $X' \in (X)^\omega - (k,X)^\omega$, then $(k \cup k', X')^0 \cap (k,X)^{(k \cup k')-k} = \phi$. Let us finally assume that $X' \notin (X)^\omega$. Then there are $i_0, i_1, j \in \omega$ such that $i_0 < i_1$, $X^{-1}(\{j\}) \cap (X')^{-1}(\{i_0\}) \neq \phi$ and $X^{-1}(\{j\}) \cap (X')^{-1}(\{i_1\}) \neq \phi$. Pick $t \in (k' \cup 1, I_{i_1})^0$ with $t(i_0) \neq t(i_1)$. Then $(k' \cup 1, A(t,X'))^0 \cap (k,X)^{(k' \cup 1)-k} = \phi$.

■

THEOREM 7.6.- $(k,X)^\omega \in \mathcal{C}^\omega_{\mathcal{F}}$ *for every* $(k,X) \in P_{(\omega)^\omega}$.

Proof – By Theorem 7.1 (i) and Lemma 7.5.

■

Let us recall the following definition. Let $R_x \subseteq (\omega)^\omega$ for $x \in \bigcup_{n \in \omega} \omega^n$. Then the result of applying the *operation* $\mathcal{A}$ to the R_x's is the set $\bigcup_{h \in \omega^\omega} \bigcap_{i \in \omega} R_{h \restriction i}$.

THEOREM 7.7.- $\mathcal{C}^\omega_{\mathcal{F}}$ *is closed under the operation* $\mathcal{A}$.

Proof – Let $R_x \in \mathcal{C}^\omega_{\mathcal{F}}$ for $x \in \bigcup_{n \in \omega} \omega^n$. Given $x \in \omega^n$, where $n \in \omega$, we put $W^x = \bigcup_{h \supset x} \bigcap_{i \in \omega} R_{h \restriction i}$ and for every $j \in \omega$, $W^x_j = W^{x \cup \{(n,j)\}}$. Notice that $W^x \subseteq R_x$ and $W^x = \bigcup_{j \in \omega} W^x_j$.

Fix $k \in \omega$ and $X \in \mathcal{F}$. Let us assume that $W^\phi \cap (k,T)^\omega \neq \phi$ for all $T \in \mathcal{F} \cap (k,X)^\omega$. Then there exists $S_0 \in \mathcal{F} \cap (k,X)^\omega$ such that $(k,S_0)^\omega \subseteq R_\phi$. We are going to define a winning strategy τ for II in $G_{\mathcal{F}}(k, S_0, (\omega)^\omega - W^\phi)$. Let I's successive moves be $Y_0, Y_1, \ldots$

By Lemmas 5.1 (i) and 5.5, II has a winning strategy τ_0 in $G_{\mathcal{F}}(k, S_0, \mathcal{F}^\omega - \mathcal{W}_0)$, where $\mathcal{W}_0$ consist of all $(Z_0, Z_1, \ldots) \in \mathcal{F}^\omega$ with the property that there exists $j \in \omega$ such that $W^\phi_j \cap (k+1+j, T)^\omega \neq \phi$ for all $T \in \mathcal{F} \cap (k+1+j, Z_j)^\omega$. Set $Z^0_0 = \tau_0(Y_0)$, $Z^0_1 = \tau_0(Y_0, Y_1)$, etc. Now pick $j_0 \in \omega$ so that $W^\phi_{j_0} \cap (k+1+j_0, T)^\omega \neq \phi$ for all $T \in \mathcal{F} \cap (k+1+j_0, Z^0_{j_0})^\omega$. Put $x_1 = \{(0, j_0)\}$ and pick $S_1 \in \mathcal{F} \cap (k+1+j_0, Z^0_{j_0})^\omega$ so that $(k+1+j_0, S_1)^\omega \subseteq R_{x_1}$. We put $\tau(Y_0, \ldots, Y_i) = Z^0_i$ for every $i < j_0$, and $\tau(Y_0, \ldots, Y_{j_0}) = S_1$.

By Lemmas 5.1 (i) and 5.5, II has a winning strategy τ_1 in $G_{\mathcal{F}}(k+1+j_0, S_1, \mathcal{F}^\omega - \mathcal{W}_1)$, where $\mathcal{W}_1$ consists of all $(Z_0, Z_1, \ldots) \in \mathcal{F}^\omega$ with the property that there exists $j \in \omega$ such that $W_j^{x_1} \cap (k+2+j_0+j, T)^\omega \neq \phi$ for all $T \in \mathcal{F} \cap (k+2+j_0+j, Z_j)^\omega$. Set $Z_0^1 = \tau_1(Y_{j_0+1})$, $Z_1^1 = \tau_1(Y_{j_0+1}, Y_{j_0+2})$, etc. Now pick $j_1 \in \omega$ so that $W_{j_1}^{x_1} \cap (k+2+j_0+j_1, T)^\omega \neq \phi$ for all $T \in \mathcal{F} \cap (k+2+j_0+j_1, Z_{j_1}^1)^\omega$. Put $x_2 = x_1 \cup \{(1, j_1)\}$, and pick $S_2 \in \mathcal{F} \cap (k+2+j_0+j_1, Z_{j_1}^1)^\omega$ so that $(k+2+j_0+j_1, S_2)^\omega \subseteq R_{x_2}$. We put $\tau(Y_0, \ldots, Y_{j_0+1+i}) = Z_i^1$ for every $i < j_1$, and $\tau(Y_0, \ldots, Y_{j_0+1+j_1}) = S_2$, etc.

Thus $G_{\mathcal{F}}(k, X, W^\phi)$ is determined for every $(k, X) \in P_{\mathcal{F}}$, and therefore $W^\phi \in \mathcal{C}_{\mathcal{F}}^\omega$ by Proposition 7.4. ∎

Let us mention one last property of $\mathcal{N}_{\mathcal{F}}^\omega$.

PROPOSITION 7.8.- *Let $T \in (\omega)^\omega$ be such that $X \notin (T)^\omega$ for all $X \in \mathcal{F}$. Then $\{Y \in (\omega)^\omega : Y \sqcap T \notin (\omega)^1\} \in \mathcal{N}_{\mathcal{F}}^\omega$.*

Proof – Fix $(k, X) \in P_{\mathcal{F}}$. As $(0, T)^\omega \in \mathcal{N}_{\mathcal{F}}^\omega$ by Theorem 7.6, there exists $Y \in \mathcal{F} \cap (k, X)^\omega$ such that $(T)^\omega \cap (0, Y)^\omega = \phi$. Define $S \in (\omega)^\omega$ by letting

$$\mathcal{B}(S) = \{X^{-1}(\{i\}) : i \leq k\} \cup (\{Y^{-1}(\{j\}) - \big(\bigcup_{i<k} X^{-1}(\{i\})\big) : j \in \omega\} - \{\phi\}).$$

Then $S \in \mathcal{F} \cap (k, X)^\omega$. Moreover, $T \cap S \in (\omega)^m$ for some $m \in \omega$. If $m = 1$, then clearly $(k, S)^\omega \subseteq \{Z \in (\omega)^\omega : T \cap Z \in (\omega)^1\}$. Let us now suppose that $m > 1$. Pick $j_d \in \omega$ for $d < m$ so that $S^{-1}(\{j_d\}) \subseteq (T \sqcap S)^{-1}(\{d\})$. Put $q = k \cup \big(\bigcup_{d<m} j_d\big)$. Clearly, there exists $r < m$ with the property that $S^{-1}(\{j\}) \subseteq (T \sqcap S)^{-1}(\{r\})$ for infinitely many $j \in \omega$. Pick $e_0, e_1, \ldots, e_{m-1}$ so that $q < e_0 < e_1 < \cdots < e_{m-1} < \omega$ and for all $d < m$, $S^{-1}(\{e_d\}) \subseteq (T \sqcap S)^{-1}(\{r\})$. Now select $t \in \big(q, I_{e_{m-1}}\big)^0$ so that $e_d \in t^{-1}(\{j_d\})$ for all $d < m$, and set $Z = A(t, S)$. Then $Z \in \mathcal{F} \cap (k, X)^\omega$. Moreover, we have that $(k, Z)^\omega \subseteq \{R \in (\omega)^\omega : T \sqcap R \in (\omega)^1\}$. ∎

Assume that $\mathcal{F}$ has the property that for any two members X, X' of $\mathcal{F}$, either $X \sqcap X' \in \mathcal{F}$ or $X \sqcap X' \notin (\omega)^\omega$. Then we can put a topology on $(\omega)^\omega$ by taking as basic open sets ϕ and all $(k, X)^\omega$ for $(k, X) \in P_{\mathcal{F}}$.

THEOREM 7.9.- *Assume that given $X, X' \in \mathcal{F}$, either $X \sqcap X' \in \mathcal{F}$ or $X \sqcap X' \notin (\omega)^\omega$. Then $\mathcal{C}_{\mathcal{F}}^\omega$ is the set of all $W \subseteq (\omega)^\omega$ which have the Baire property.*

Proof – By Lemma 6.3 and Theorem 7.1 ((i) and (iii)) ∎

Observe that the Dual Ellentuck Theorem follows from Theorem 7.9 since the assumption of Theorem 7.9 trivially holds if $\mathcal{F} = (\omega)^\omega$.

8. Generalizing the Dual Ramsey Theorem

The purpose of this section is to obtain a satisfactory generalization of the Dual Ramsey Theorem. The reader might wonder why we should devote a whole section to the subject. Doesn't the Dual Ramsey Theorem follow from the Dual Ellentuck Theorem, just as Ramsey's Theorem does from Ellentuck's Theorem ? Some words of explanation are in order, especially since the terminology is somewhat misleading. The Dual Ellentuck Theorem deals with partitions of ω into infinitely many blocks, whereas the Dual Ramsey Theorem is an infinite collection of statements, each one of which deals with partitions of ω into a fixed finite number of blocks. The Dual Ellentuck is an "if and only if" statement and has thus reached its final form, unlike the Dual Ramsey Theorem which is only an "if" statement. There are actually two versions of the Dual Ramsey Theorem, which are both stated in topological terms. The first one, which is Theorem 1.2 in [2], does follow from the Dual Ellentuck Theorem. The second one, which is due to Prömel and Voigt [18] strengthens the result of Carlson and Simpson. It is not clear to us whether it can be easily deduced from the Dual Ellentuck Theorem. Moreover, each one of these two versions was obtained as a special case of an "alphabetized" result (respectively, Theorem 2.2 in [2], which again follows from the Dual Ellentuck Theorem, and Theorem B in [18]).

As for us, we forsake the topological approach and do it the Mathias way, which works fine. Our results generalize the Dual Ramsey Theorem in the sense that Theorem B of [18] can be deduced from them. They are however incomplete inasmuch as we have been unable to show closure under operation $\mathcal{A}$ (except for the special case when the game-family is in fact a game-filter, see Section 10).

Throughout the remainder of this section $\mathcal{F}$ and n will denote, respectively, a fixed game-family and a fixed element of $\omega - \{0,1\}$.

We set $P^n_{\mathcal{F}} = \{(k,X) \in P_{\mathcal{F}} : k < n\}$.

We let $\mathcal{N}^n_{\mathcal{F}}$ (respectively, $\mathcal{C}^n_{\mathcal{F}}$) consist of all $W \subseteq (\omega)^n$ with the property that for all $(k,X) \in P^n_{\mathcal{F}}$, there exists $Z \in \mathcal{F} \cap (k,X)^\omega$ such that $W \cap (k,Z)^{n-1-k} = \phi$ (respectively, such that either $(k,Z)^{n-1-k} \subseteq W$ or $W \cap (k,Z)^{n-1-k} = \phi$).

We let $\overline{\mathcal{N}}^n_{\mathcal{F}}$ (respectively, $\overline{\mathcal{C}}^n_{\mathcal{F}}$) consist of all $W \subseteq (\omega)^n$ with the property that the set of all $(k,X) \in P^n_{\mathcal{F}}$ such that $W \cap (k,X)^{n-1-k} = \phi$ (respectively, such that either $(k,X)^{n-1-k} \subseteq W$ or $W \cap (k,X)^{n-1-k} = \phi$) is dense in $P^n_{\mathcal{F}}$.

LEMMA 8.1.- *Let Λ be a dense open subset of $P^n_{\mathcal{F}}$, and let $d < n$ and $(k,X) \in P_{\mathcal{F}}$ with $k \geq n-1$. Then there exists $Y \in \mathcal{F} \cap (k,X)^\omega$ such that $\big(n-1, A(t,Y)\big) \in \Lambda$ for every $t \in (d, I_k)^{n-1-d}$.*

Proof – Let t_j for $j \leq q$ be an enumeration of the elements of $(d, I_k)^{n-1-d}$. Define Z_j, T_j and Y_j for $j \leq q$ so that

(0) $Z_0 = A(t_0, X)$;

(1) $Z_{i+1} = A(t_{i+1}, Y_i)$;

(2) $T_j \in \mathcal{F} \cap (n-1, Z_j)^\omega$;

(3) $(n-1, t_j) \in \Lambda$;

(4) $Y_0 = D(t_0, X, T_0)$;

(5) $Y_{i+1} = D(t_{i+1}, Y_i, T_i)$.

Then setting $Y = Y_q$, we have that Y is as desired. ∎

LEMMA 8.2.- *Let Λ be a dense open subset of $P^n_{\mathcal{F}}$, and let $(k, X) \in P^n_{\mathcal{F}}$. Then there exists $T \in \mathcal{F} \cap (k, X)^\omega$ such that $\big(n-1, A(t,T)\big) \in \Lambda$ for every $t \in \bigcup_{n-1 \leq j < \omega} (k, I_j)^{n-1-k}$.*

Proof – Let W consist of all $T \in \mathcal{F} \cap (k, X)^\omega$ such that $\big(n-1, A(t,T)\big) \in \Lambda$ for every $t \in \bigcup_{n-1 \leq j < \omega} (k, I_j)^{n-1-k}$. We define a winning strategy τ for II in $G_{\mathcal{F}}(k, X, \mathcal{F} - W)$ as follows. Let I successively play $Y_0, Y_1, \ldots$ If $i < n-1-k$, we put $\tau(Y_0, \ldots, Y_i) = Y_i$. Otherwise, we use Lemma 8.1 to find $Z_i \in \mathcal{F} \cap (k+i, Y_i)^\omega$ such that $\big(n-1, A(t, Z_i)\big) \in \Lambda$ for every $t \in (k, I_{k+i})^{n-1-k}$, and we put $\tau(Y_0, \ldots, Y_i) = Z_i$. It follows from Lemma 3.1 that $W \neq \phi$. ∎

THEOREM 8.3.- *The following hold :*

(i) $\mathcal{C}^n_{\mathcal{N}} = \overline{\mathcal{C}}^n_{\mathcal{F}}$.

(ii) $\mathcal{N}^n_{\mathcal{F}} = \overline{\mathcal{N}}^n_{\mathcal{F}}$.

(iii) *$\overline{\mathcal{N}}^n_{\mathcal{F}}$ is closed under countable unions.*

Proof – (i) : It is immediate that $\mathcal{C}^n_{\mathcal{F}} \subseteq \overline{\mathcal{C}}^n_{\mathcal{F}}$. To show the reverse inclusion, fix $W \in \overline{\mathcal{C}}^n_{\mathcal{F}}$ and $(k, X) \in P^n_{\mathcal{F}}$. By Lemma 8.2 there exists $T \in \mathcal{F} \cap (k, X)^\omega$ such that for every $t \in \bigcup_{n-1 \leq j < \omega} (k, I_j)^{n-1-k}$, either $\big(n-1, A(t,T)\big)^0 \subseteq W$ or $W \cap \big(n-1, A(t,T)\big)^0 = \phi$. Let us assume that $W \cap (k, Y)^{n-1-k} \neq \phi$ for every $Y \in \mathcal{F} \cap (k, T)^\omega$. Then by Lemmas 5.1 (iii) and 5.2, there exists $Z \in \mathcal{F} \cap (k, T)^\omega$ such that $W \cap (n-1, Z')^0 \neq \phi$ for every $Z' \in \mathcal{F} \cap (k, Z)^\omega$. It is easy to see that $(k, Z)^{n-1-k} \subseteq W$.

(ii) : By Lemma 8.2.

(iii) : Let $W_i \in \overline{\mathcal{N}}^n_{\mathcal{F}}$ for each $i \in \omega$.

Set

$$\Lambda_i = \{(k, X) \in P_{\mathcal{F}} : \big(k \cap (n-1), X\big)^{(n-1)-(k \cap (n-1))} \cap W_i = \phi\}$$

for every $i \in \omega$. Each Λ_i is inductive, by Lemma 5.1 (iii), and dense, by Lemma 8.1. Hence $\bigcap_{i\in\omega} \Lambda_i$ is inductive and dense by Proposition 5.6. It now follows from Proposition 5.4 that $\bigcup_{i\in\omega} W_i \in \mathcal{N}^n_{\mathcal{F}}$. ∎

THEOREM 8.4.- $(k,X)^{n-1-k} \in \mathcal{C}^n_{\mathcal{F}}$ *for every* $(k,X) \in P^n_{(\omega)^\omega}$.

Proof – By Theorem 8.3 (i) and Lemma 7.5. ∎

PROPOSITION 8.5.- $\mathcal{C}^n_{\mathcal{F}}$ *is closed under countable unions.*

Proof – By Theorem 8.3 (i), it suffices to show that any countable union of members of $\mathcal{C}^n_{\mathcal{F}}$ lies in $\overline{\mathcal{C}}^n_{\mathcal{F}}$. Thus let $W_i \in \mathcal{C}^n_{\mathcal{F}}$ for $i \in \omega$, and let $(k,X) \in P^n_{\mathcal{F}}$. Set

$$\Lambda_i = \{(k',X') \in P_{\mathcal{F}} : \big(k' \cap (n-1), X'\big)^{(n-1)-(k'\cap(n-1))} \cap W_i = \phi\}.$$

for every $i \in \omega$. Assume that $(k,Z) \notin \bigcap_{i\in\omega} \Lambda_i$ for every $Z \in \mathcal{F} \cap (k,X)^\omega$. Then by Lemmas 5.1 (iii) and 5.5, there exist $T \in \mathcal{F} \cap (k,X)^\omega$ and $i \in \omega$ such that $(k+1+i, S) \notin \Lambda_i$ for all $S \in \mathcal{F} \cap (k+1+i, T)^\omega$. Let us first assume that $k+1+i < n$. Then there exists $S \in \mathcal{F} \cap (k+1+i, T)^\omega$ such that either $(k+1+i,S)^{n-1-(k+1+i)} \subseteq W_i$ or $(k+1+i,S)^{n-1-(k+1+i)} \cap W_i = \phi$. Clearly, $(k+1+i,S) \leq (k,X)$. Moreover $(k+1+i,S) \notin \Lambda_i$, and consequently $(k+1+i,S)^{n-1-(k+1+i)} \subseteq \bigcup_{i\in\omega} W_i$. Assume now that $k+1+i \geq n$. By Lemma 8.1, there exists $Y \in \mathcal{F} \cap (k+1+i,T)^\omega$ such that for every $t \in (k, I_{k+1+i})^{n-1-k}$, either $\big(n-1, A(t,Y)\big)^0 \subseteq W_i$ or $\big(n-1, A(t,Y)\big)^0 \cap W_i = \phi$. There exists $u \in (k, I_{k+1+i})^{n-1-k}$ such that $\big(n-1, A(u,Y)\big)^0 \subseteq W_i$, since otherwise we would have $(n-1,Y)^0 \cap W_i = \phi$ and therefore $(k+1+i,Y) \in \Lambda_i$, a contradiction. Clearly, $\big(n-1, A(u,Y)\big) \leq (k,X)$ and $\big(n-1, A(u,Y)\big)^0 \subseteq \bigcup_{i\in\omega} W_i$. ∎

Finally, let us see how the Prmel-Voigt version of the Dual Ramsey Theorem can be derived from the results of this section.

We topologize $(\omega)^n$ by taking as basic open sets ϕ and all sets of the form $\{X \in (\omega)^n : t \subseteq X\}$, where $t \in \bigcup_{m\in\omega} (m)^n$.

The following is readily checked.

LEMMA 8.6.- $\{X \in (\omega)^n : t \subseteq X\} \in \mathcal{C}^n_{(\omega)^\omega}$ *for every* $t \in \bigcup_{m\in\omega} (m)^n$.

The following is due to Prmel and Voigt (see Theorem G in [18]).

LEMMA 8.7.- *Let $k < n$, and let W be a meager subset of $(\omega)^n$. Then there is $X \in (\omega)^\omega$ such that $W \cap (k, X)^{n-1-k} = \phi$.*

Proof – Select U_j for $j \in \omega$ so that each U_j is a dense open subset of $(\omega)^\omega$ and $W \cap (\bigcap_{j\in\omega} U_j) = \phi$. Define $X_0, X_1, X_2, \ldots \in (\omega)^\omega$ as follows :
Set $X_0 = I_\omega$.
Pick $u \in \bigcup_{n-1\leq i} (n-1, I_i)^0$ so that $(n-1, A(u, X_0))^0 \subseteq U_0$. Set $X_1 = A(u, X_0)$.
Let t_d for $d \leq q$ be an enumeration of the elements of $(k, I_n)^{n-1-k}$. Define S_d, Y_d, Z_d and u_d for $d \leq q$ so that

(0) $Z_0 = X_1$;

(1) $S_d = A(t_d, Z_d)$;

(2) $u_d \in \bigcup_{n-1\leq i} (n-1, I_i)^0$;

(3) $Y_d = A(u_d, S_d)$;

(4) $(n-1, Y_d)^0 \subseteq U_0 \cap U_1$;

(5) $Z_{d+1} = D(t_d, Z_d, Y_d)$ if $d < q$.

Then set $X_2 = D(t_q, Z_q, Y_d)$. etc.
Finally, $X = \bigsqcap_{j\in\omega} X_j$ is as desired. ■

The following is Theorem B in [18].

PROPOSITION 8.8.- *Suppose that $k < n$ and $0 < r < \omega$, and let $F : (\omega)^n \to r$ be such that $F^{-1}(\{i\})$ has the Baire property for each $i < r$. Then there is $Z \in (\omega)^\omega$ such that F is constant on $(k, Z)^{n-1-k}$.*

Proof – For $i < r$, select an open subset U_i of $(\omega)^n$ such that $U_i \Delta F^{-1}(\{i\})$ is meager. By Lemma 8.7, there is $X \in (\omega)^\omega$ such that

$$\Big(\bigcup_{i<r} (U_i \Delta F^{-1}(\{i\}))\Big) \bigcap (k, X)^{n-1-k} = \phi.$$

By Proposition 8.5 and Lemma 8.6, there is $Z \in (k, X)^\omega$ such that for each $i < r$, either $(k, Z)^{n-1-k} \subseteq U_i$ (and hence $(k, Z)^{n-1-k} \subseteq F^{-1}(\{i\})$), or $U_i \bigcap (k, Z)^{n-1-k} = \phi$ (and hence $F^{-1}(\{i\}) \bigcap (k, Z)^{n-1-k} = \phi$). Clearly, $(k, Z)^{n-1-k} \subseteq F^{-1}(\{j\})$ for some $j < r$. ■

9. The segment-coloring property

$\mathcal{F} \subseteq (\omega)^\omega$ has the *segment-coloring property* if given $n \in \omega$, $X \in \mathcal{F}$ and $F : \bigcup_{m,p\in\omega} (p)^m \to 2$, there is $Z \in \mathcal{F} \cap (X)^\omega$ such that for every $k \in \omega$, F is constant on $< Z >_k^n$.

In this short section we show that every game-family has the segment-coloring property. We will need the following generalization of Lemma 4.1.

PROPOSITION 9.1.- *Suppose that* $\mathcal{F}$ *is a game-family, and let* $n, k \in \omega, X \in \mathcal{F}$ *and* $\mathcal{D} \subseteq < X >_k^n$. *Then there exists* $S \in \mathcal{F} \cap (k, X)^\omega$ *such that either* $< S >_k^n \subseteq \mathcal{D}$ *or else* $< S >_k^n \cap \mathcal{D} = \phi$.

Proof – This is a consequence of Theorem 7.1 (i) since clearly

$$\{Y \in (\omega)^\omega : s(Y, k+1+n) \in \mathcal{D}\} \in \overline{\mathcal{C}}_{\mathcal{F}}^\omega.$$

∎

Proposition 9.1 generalizes Proposition 7.7 of [8].

PROPOSITION 9.2.- *Suppose that* $\mathcal{F}$ *is a game-family. Then* $\mathcal{F}$ *has the segment-coloring property.*

Proof – Fix $n \in \omega$, $X \in \mathcal{F}$ and $F : \bigcup_{m,p\in\omega} (p)^m \to 2$. Let $\mathcal{W}$ be the set of all $(Z_0, Z_1, \ldots) \in \mathcal{F}^\omega$ such that for every $i \in \omega$, F is constant on $< Z_i >_i^n$. It easily follows from Proposition 9.1 that II has a winning strategy in $G_{\mathcal{F}}(0, X, \mathcal{F}^\omega - \mathcal{W})$. Since II has no winning strategy in $G_{\mathcal{F}}(0, X, \mathcal{F})$, one can find $Z_0 \in (X)^\omega, Z_1 \in (1, Z_0)^\omega, Z_2 \in (2, Z_1)^\omega, \ldots$ so that $(Z_0, Z_1, \ldots) \in \mathcal{W}$ and $\bigsqcap_{i\in\omega} Z_i \in \mathcal{F}$. Given $k \in \omega$, $<\bigsqcap_{i\in\omega} Z_i >_k^n \subseteq < Z_k >_k^n$ and hence F is constant on $<\bigsqcap_{i\in\omega} Z_i >_k^n$. ∎

10. Game-filters

A *filter on* $(\omega)^\omega$ is a subset $\mathcal{F}$ of $(\omega)^\omega$ such that (a) $X \sqcap Y \in \mathcal{F}$ for all $X, Y \in \mathcal{F}$, and (b) $\{Y \in (\omega)^\omega : X \leq Y\} \subseteq \mathcal{F}$ for all $X \in \mathcal{F}$.

By a *game-filter* we mean a game-family which is a filter on $(\omega)^\omega$.

This section is concerned with existence and properties of game-filters.

Let us first remark that every game-family is associated with a filter on $(\omega)^\omega$. We set $\mathcal{F}_* = \{Z \in (\omega)^\omega : (\forall X \in \mathcal{F})\ Z \sqcap X \in (\omega)^\omega\}$ for every $\mathcal{F} \subseteq (\omega)^\omega$.

Lemma 10.1.- *If $\mathcal{F}$ is a game-family, then*

$$\mathcal{F}_* = \{Z \in (\omega)^\omega : (\forall X \in \mathcal{F})(\exists Y \in \mathcal{F})\ Y \leq Z \sqcap X\}.$$

Proof – Given $Z \in \mathcal{F}_*$ and $X \in \mathcal{F}$, there exists by Theorem 7.6 $Y \in \mathcal{F} \cap (X)^\omega$ such that either $(Y)^\omega \subseteq (Z \sqcap X)^\omega$ or $(Y)^\omega \cap (Z \sqcap X)^\omega = \phi$. We have that $Z \sqcap Y \in (Y)^\omega \cap (Z \sqcap X)^\omega$, and therefore $Y \in (Z \sqcap X)^\omega$. ∎

Proposition 10.2.- *If $\mathcal{F}$ is a game-family, then $\mathcal{F}_*$ is a filter on $(\omega)^\omega$.*

Proof – Let $Z_0, Z_1 \in \mathcal{F}_*$. Given $X \in \mathcal{F}$, there is by Lemma 10.1 $Y \in \mathcal{F}$ with $Y \leq Z_0 \sqcap X$. We have that $Z_1 \sqcap Y \in (\omega)^\omega$. Hence $(Z_0 \sqcap Z_1) \sqcap X \in (\omega)^\omega$. Thus $Z_0 \sqcap Z_1 \in \mathcal{F}_*$. ∎

Let us make the following remark. Given a game-family $\mathcal{F}$, set $\overline{\mathcal{F}} = \{Y \in (\omega)^\omega : (\exists X \in \mathcal{F}) X \leq Y\}$. Then $\overline{\mathcal{F}}$ is also a game-family. Moreover $\mathcal{F} \subseteq \overline{\mathcal{F}}$, $\mathcal{F}_* = (\overline{\mathcal{F}})_*$ and $\overline{\overline{\mathcal{F}}} = \overline{\mathcal{F}}$. We do not know whether $\overline{\mathcal{F}}$ can be recovered from $\mathcal{F}_*$, i.e. whether $\{Y \in (\omega)^\omega : (\forall Z \in \mathcal{F}_*) Z \sqcap Y \in (\omega)^\omega\} \subseteq \overline{\mathcal{F}}$ (notice that the reverse inclusion is obvious).

Game-filters have attractive combinatorial properties. The Dual Ellentuck Theorem can be relativized to game-filters in two different ways (see Theorem 6.4 and the remark at the end of Section 7). The Dual Ramsey Theorem also admits a nice formulation in this context. Let $\mathcal{F}$ be a game-filter, and let $n \in \omega - \{0, 1\}$. Then we can put a topology on $(\omega)^n$ by taking as basic open sets ϕ and all $(k, X)^{n-1-k}$ for $(k, X) \in P^n_{\mathcal{F}}$. Moreover by Lemma 6.3 and Theorem 8.3 ((i) and (iii)), $\mathcal{C}^n_{\mathcal{F}}$ is the set of all $W \subseteq (\omega)^n$ which have the Baire property with respect to this topology. Let us further remark that by Proposition 7.2 of [13], every game-filter is maximal (as a filter on $(\omega)^\omega$).

We will now see that game-filters can be added by forcing. Let us start with a few definitions.

Let $X, Y \in (\omega)^\omega$. We write $X \leq^* Y$ just in case $A(t, X) \leq Y$ for some $t \in \bigcup_{i \in \omega} (I_i)^1$.

We write $X \sim Y$ to mean that $X \leq^* Y$ and $Y \leq^* X$.

We put $\dot{X} = \{Z \in (\omega)^\omega : X \sim Z\}$.

We let $\dot{X} \preccurlyeq \dot{Y}$ just in case $X \leq^* Y$.

We set $Q_{\mathcal{F}} = \{\dot{X} : X \in \mathcal{F}\}$ for all $\mathcal{F} \subseteq (\omega)^\omega$.

Lemma 10.3.- *Let $\mathcal{F}$ be a game-family. Then $(Q_{\mathcal{F}}, \preccurlyeq)$ is σ-distributive.*

Proof – Let E_i be a dense open subset of $Q_{\mathcal{F}}$ for each $i \in \omega$. To show that $\bigcap_{i\in\omega} E_i$ is dense, let $\dot{X} \in Q_{\mathcal{F}}$ be arbitrary. We are going to define a winning strategy τ for II in $G_{\mathcal{F}}(0, X, \mathcal{F}^\omega - \mathcal{W})$, where $\mathcal{W}$ consists of all $(Z_0, Z_1, \ldots) \in \mathcal{F}^\omega$ such that $\dot{Z}_i \in E_i$ for every $i \in \omega$. To define $\tau(Y_0, \ldots, Y_i)$, pick $T_i \in \mathcal{F}$ so that $\dot{T}_i \in E_i$ and $\dot{T}_i \preccurlyeq \dot{Y}_i$. Then select $Z_i \in \dot{T}_i \cap (i, Y_i)^\omega$, and put $\tau(Y_0, \ldots, Y_i) = Z_i$. There are $Y_0 \in \mathcal{F} \cap (0, X)^\omega$, $Y_1 \in \mathcal{F} \cap \big(1, \tau(Y_0)\big)^\omega$, $Y_2 \in \mathcal{F} \cap \big(2, \tau(Y_0, Y_1)\big)^\omega, \ldots$ such that $\bigsqcap_{i\in\omega} \tau(Y_0, \ldots, Y_i) \in \mathcal{F}$. Then setting $S = \bigsqcap_{i\in\omega} \tau(Y_0, \ldots, Y_i)$, we have that $\dot{S} \preccurlyeq \dot{X}$ and $\dot{S} \in \bigcap_{i\in\omega} E_i$. ■

Thus (see Lemma 19.6 in [11]) assuming $\mathcal{F}$ is a game-family, forcing with $(Q_{\mathcal{F}}, \preccurlyeq)$ adds no new functions from ω into V.

Let $\mathcal{F}$ be a game-family, and let H be $Q_{\mathcal{F}}$-generic over V. Then in $V[H]$, we set

$$\mathcal{E}_H = \{X \in (\omega)^\omega : (\exists \dot{Y} \in H) \dot{Y} \preccurlyeq \dot{X}\}.$$

The following generalizes Theorem 5.1 in [6].

Theorem 10.4.- *Let $\mathcal{F}$ be a game-family, and let H be $Q_{\mathcal{F}}$-generic over V. Then in $V[H]$, $\mathcal{F}_* \subseteq \mathcal{E}_H$ and $\mathcal{E}_H$ is a game-filter.*

Proof – We have by Lemma 10.1 that $\{\dot{Y} \in Q_{\mathcal{F}} : \dot{Y} \preccurlyeq \dot{Z}\}$ is dense in $Q_{\mathcal{F}}$ for every $Z \in \mathcal{F}_*$, which proves the first assertion. Let us now turn to the second assertion. We clearly have that $\mathcal{E}_H$ is a filter on $(\omega)^\omega$. Moreover, $\dot{X} \subseteq \mathcal{E}_H$ for every $X \in \mathcal{E}_H$. To complete the proof, it suffices to show that II has no winning strategy in $G_{\mathcal{H}_H}(0, I_\omega, \mathcal{H}_H)$, where $\mathcal{H}_H = \bigcup H$.

Let Γ be the canonical name for the generic object. Let $\dot{X} \in Q_{\mathcal{F}}$ and a name ρ be such that

$$\dot{X} \Vdash \rho \text{ is a strategy for II in } G_{\mathcal{H}_\Gamma}(\check{0}, \check{I}_\omega, \mathcal{H}_\Gamma).$$

We will define (in V) a strategy τ for II in $G_{\mathcal{F}}(0, X, \phi)$. To define $\tau(Y_0, \ldots, Y_i)$, we proceed as follows. There exist $\dot{R}_i \in P_{\mathcal{F}}$ and $Z_i \in (\omega)^\omega$ such that

$\dot{R}_i \preccurlyeq \dot{Y}_i$ and $\dot{R}_i \Vdash \rho(\check{Y}_0, \dots, \check{Y}_i) = \check{Z}_i$. We use Theorem 7.6 to find $S_i \in \mathcal{F} \cap (R_i)^\omega$ such that either $(S_i)^\omega \subseteq (Z_i)^\omega$ or $(S_i)^\omega \cap (Z_i)^\omega = \phi$. As $\dot{S}_i \Vdash (\dot{\check{Z}}_i) \in \Gamma$, we have that $S_i \le Z_i$. We pick $T_i \in \dot{S}_i \cap (1+i, Z_i)^\omega$ and put $\tau(Y_0, \dots, Y_i) = T_i$. Notice that $\bigsqcap_{i\in\omega} T_i = \bigsqcap_{i\in\omega} Z_i$. There are $Y_0 \in \mathcal{F} \cap (0, X)^\omega$, $Y_1 \in \mathcal{F} \cap \big(1, \tau(Y_0)\big)^\omega$, $Y_2 \in \mathcal{F} \cap \big(2, \tau(Y_0, Y_1)\big)^\omega$, $\cdots$ with $\bigsqcap_{i\in\omega} \tau(Y_0, \dots, Y_i) \in \mathcal{F}$. Setting $T = \bigsqcap_{i\in\omega} \tau(Y_0, \dots, Y_i)$, we have that $\dot{T} \Vdash (\dot{\check{T}}) \in \mathcal{H}_\Gamma$, and therefore $\dot{T} \Vdash \bigsqcap_{i\in\check{\omega}} \rho(\check{Y}_0, \dots, \check{Y}_i) \in \mathcal{H}_\Gamma$. Hence

$$\dot{T} \Vdash \rho \text{ is not a winning strategy for II in } G_{\mathcal{H}_\Gamma}(\check{O}, \check{I}_\omega, \mathcal{H}_\Gamma).$$

∎

We do not know whether CH implies the existence of game-filters. On the other hand, it is shown in [7] that under CH there are many filters on $(\omega)^\omega$ with the segment-coloring property. See [8] and [13] for more constructions of nice filters on $(\omega)^\omega$ under CH.

We will finally use an argument of Di Prisco and Henle [3] to show that the existence of a game-filter is compatible with a partition property which contradicts the Axiom of Choice.

For $T \in (\omega)^\omega$ and $k \in \omega - \{0\}$, we set

$$O_T^k = \{X \in (\omega)^\omega : (\forall i \in \{1, 2, \dots, k\}) X^{-1}(\{i\}) = T^{-1}(\{i\})\}.$$

We put a topology on $(\omega)^\omega$ by taking as basic open sets ϕ and all O_T^k for $T \in (\omega)^\omega$ and $k \in \omega - \{0\}$.

$F : (\omega)^\omega \to 2$ is *clopen if* $F^{-1}(\{j\})$ is open for all $j < 2$.

$\omega \xleftarrow{\text{clopen}} (\omega)^\omega$ asserts the following : For every clopen $F : (\omega)^\omega \to 2$, there exists $X \in (\omega)^\omega$ such that F is constant on $(X)^\omega$.

For β with $2 \le \beta \le \omega$, $\omega \longleftarrow (\omega)^\beta$ asserts that for every $F : (\omega)^\beta \to 2$, there exists $X \in (\omega)^\omega$ such that F is constant on $(X)^\beta$.

As was shown in [2], $\omega \longleftarrow (\omega)^2$ does not hold under the Axiom of Choice.

It is not difficult to check that $\omega \xleftarrow{\text{clopen}} (\omega)^\omega$ implies $\omega \longleftarrow (\omega)^n$ for every n with $2 \le n < \omega$.

Concerning the hypothesis of the following proposition, let us recall that it was shown in [2] that if ZFC + "there exists an inaccessible cardinal" is consistent, then so is ZF+DC+ $\omega \longleftarrow (\omega)^\omega$.

THEOREM 10.5.- *Assume that* DC + $\omega \longleftarrow (\omega)^\omega$ *holds in* V*, and let* H *be* $Q_{(\omega)^\omega}$ *- generic over* V*. Then* $\omega \overset{clopen}{\longleftarrow} (\omega)^\omega$ *holds in* $V[H]$.

Proof – Let $\dot{X} \in Q_{(\omega)^\omega}$ and a name F be such that

$$\dot{X} \Vdash F \text{ is a clopen function from } (\check{\omega})^{\check{\omega}} \text{ to } \check{2}.$$

In V, define $K : (X)^\omega \to 3$ so that for $j = 0, 1, K(Y) = j$ if and only if $\dot{Y} \Vdash F(\check{Y}) = \check{j}$. Pick $Y \in (X)^\omega$ with K being constant on $(Y)^\omega$. There are $Y' \in (Y)^\omega, j \in 2, T \in (\omega)^\omega$ and $k \in \omega - \{0\}$ such that

$$\dot{Y}' \Vdash \left(\check{Y} \in O_{\check{T}}^{\check{k}} \text{ and } O_{\check{T}}^{\check{k}} \subseteq F^{-1}(\{\check{j}\})\right).$$

Define $Y'' \in (Y)^\omega$ by letting

$$\mathcal{B}(Y'') = \left\{Y^{-1}(\{i\}) : i \in \{1, \dots, k\}\right\} \bigcup \Big(\{ ((Y')^{-1}(\{m\})) - \big(\bigcup_{i \in \{1,\dots,k\}} Y^{-1}(\{i\})\big) : m \in \omega\Big\} - \{\phi\}\Big).$$

Then clearly $K(Y'') = j$. Hence K is identically j on $(Y)^\omega$.

Define $Z, S \in (Y)^\omega$ by letting

$$\mathcal{B}(Z) - \{Z^{-1}(\{0\})\} = \{Y^{-1}(\{2n+1\}) : n \in \omega\}$$

and

$$\mathcal{B}(S) - \{S^{-1}(\{0\})\} = \left\{Y^{-1}(\{2n\}) : n \in \omega - \{0\}\right\}.$$

Let $R \in (S)^\omega$ be given. We claim that $\dot{Z} \Vdash F(\check{R}) = \check{j}$. Suppose otherwise. Then there exist $Z' \in (Z)^\omega$, $D \in (\omega)^\omega$ and $q \in \omega - \{0\}$ such that

$$\dot{Z}' \Vdash \left(\check{R} \in O_{\check{D}}^{\check{q}} \text{ and } O_{\check{D}}^{\check{q}} \subseteq F^{-1}(\{\check{1} - \check{j}\})\right).$$

Set $d = \cap\{c \in \omega : \ell(Z', c) > \ell(D, q)\}$ and define $Z'' \in (Y)^\omega$ by letting

$$\mathcal{B}(Z'') - \left\{(Z'')^{-1}(\{0\})\right\} = \left\{D^{-1}(\{i\}) : 1 \le i \le q\right\} \bigcup \left\{(Z')^{-1}(\{r\}) : r \ge d\right\}.$$

Then $\dot{Z}'' \Vdash F(\check{Z}'') = \check{1} - \check{j}$, which contradicts the fact that $K(Z'') = j$. Thus $\dot{Z} \Vdash F$ is constant on $(\check{S})^{\check{\omega}}$. ∎

The combination of Theorems 9.4 and 9.5 shows that if ZF + DC + $\omega \longleftarrow (\omega)^\omega$ is consistent, then so is ZF + $\omega \overset{\text{clopen}}{\longleftarrow} (\omega)^\omega$ + "there exists a game-filter".

Let us conclude with the following easy observation.

PROPOSITION 10.6.- *Let* $\mathcal{F}$ *be a filter on* $(\omega)^\omega$*. Then there exists* $F : (\omega)^2 \to 2$ *such that for every* $X \in \mathcal{F}$*,* F *is not constant on* $(X)^2$.

Proof – Define $\theta : (\omega)^2 \to \bigcup_{2\leq\beta\leq\omega} (\omega)^\beta$ by letting

$$\mathcal{B}(\theta(S)) = \left\{S^{-1}(\{0\})\right\} \bigcup \left\{\{m\} : m \in S^{-1}(\{1\})\right\}.$$

Then define $F : (\omega)^2 \to 2$ by letting $F(S) = 1$ if and only if $\theta(S) \in \mathcal{F}$. Given $X \in \mathcal{F}$, define $Y, Z, T \in (\omega)^2$ by letting $Y^{-1}(\{0\}) = X^{-1}(\{0\})$, $Z^{-1}(\{1\}) = X^{-1}(\{1\})$ and $T^{-1}(\{1\}) = X^{-1}(\{2\})$. Then $F(Y) = 1$. As $\theta(Z)\sqcap\theta(T) \in (\omega)^1$, we do not have that $F(Z) = F(T) = 1$. ■

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