MAKING DOUGHNUTS OF COHEN REALS

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Abstract

For $a \subseteq b \subseteq \omega$ with $b \setminus a$ infinite, the set $D = \{x \in [\omega]^\omega : a \subseteq x \subseteq b\}$ is called a *doughnut*. A set $S \subseteq [\omega]^\omega$ has the *doughnut property* $\mathscr{D}$, if it contains or is disjoint from a doughnut. It is known that not every set $S \subseteq [\omega]^\omega$ has the doughnut property, but S has the doughnut property if it has the Baire property $\mathscr{B}$ or the Ramsey property $\mathscr{R}$. In this paper it is shown that a finite support iteration of length ω_1 of Cohen forcing, starting from $\mathbf{L}$, yields a model for $\mathsf{CH} + \mathbf{\Sigma^1_2}(\mathscr{D}) + \neg\mathbf{\Sigma^1_2}(\mathscr{B}) + \neg\mathbf{\Sigma^1_2}(\mathscr{R})$.

0. Introduction

Investigating arrow partition properties, Carlos DiPrisco and James Henle introduced in [DH00] the so-called doughnut property: For a set x, let $|x|$ denote its cardinality and let $[\omega]^\omega := \{x \subseteq \omega : |x| = \omega\}$. Then, for $a \subseteq b \subseteq \omega$ with $b \setminus a \in [\omega]^\omega$, the set $D = \{x \in [\omega]^\omega : a \subseteq x \subseteq b\}$ is called a **doughnut**, or more precisely, the (a,b)-doughnut. A set $S \subseteq [\omega]^\omega$ has the **doughnut property**, denoted by $\mathscr{D}$, if it contains or is disjoint from some doughnut. A set $S \subseteq [\omega]^\omega$ has the **Ramsey property**, denoted by $\mathscr{R}$, if it contains or is disjoint from some $(\emptyset, b)$-doughnut. Hence, it is obvious that if S has the Ramsey property, then it has the doughnut property as well. Like for the Ramsey property, it is easy to show—using the Axiom of Choice—that not every set $S \subseteq [\omega]^\omega$ has the doughnut property. Moreover, in the constructible universe $\mathbf{L}$ not even every Δ^1_2-set has the doughnut property. Indeed, let $A = \big\{y \in [\omega]^\omega : \forall z \in [\omega]^\omega\big(z <_{\mathbf{L}} y \to (|y \,\triangle\, z| \textit{ is infinite})\big)\big\}$, where "$\triangle$" denotes the symmetric difference, and let $S = \big\{x \in [\omega]^\omega : \forall y \in A(|x \,\triangle\, y| \textit{ is infinite or odd})\big\}$. Since "$<_{\mathbf{L}}$" is a Δ^1_2-relation (*cf.* [Je78, Theorem 97]), the set A is a Π^1_2-set and by construction, for every $x \in [\omega]^\omega$ there is a unique $y \in A$ such that $|x \,\triangle\, y|$ is finite. Thus, for every $x \in [\omega]^\omega$ we either have $x \in S$ or $\forall y \in A(|x \,\triangle\, y| \textit{ is infinite or even})$, and since "odd" and "even" are arithmetical relations, S is a Δ^1_2-set. Now, if $x \in [\omega]^\omega$ and $n \in x$, then $x \in S$ if and only if $x \setminus \{n\} \notin S$, which implies that S does not have the doughnut property. On the other hand, one can show that if S has the **Baire property** $\mathscr{B}$ (which means there is an open set $\mathcal{O}$ such that $S \,\triangle\, \mathcal{O}$ is meager), then S has also the doughnut property (*cf.* [DH00, Proposition 2.2] or [MS80]).

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Let $\mathscr{P}$ be any property of subsets of $[\omega]^\omega$. We write $\mathbf{\Sigma^1_2}(\mathscr{P})$ if every "boldface" $\mathbf{\Sigma^1_2}$-set $S \subseteq [\omega]^\omega$ has the property $\mathscr{P}$. By the facts mentioned above we have

$$\mathbf{\Sigma^1_2}(\mathscr{B}) \implies \mathbf{\Sigma^1_2}(\mathscr{D}) \impliedby \mathbf{\Sigma^1_2}(\mathscr{R}).$$

This is similar to the fact that if all $\mathbf{\Sigma^1_2}$-sets S have the Baire property or the Ramsey property, then all $\mathbf{\Sigma^1_2}$-sets are $\boldsymbol{K_\sigma}$**-regular**, denoted by $\mathscr{K}_\sigma$. (The property $\mathscr{K}_\sigma$ will be defined later.) Thus, we also have

$$\mathbf{\Sigma^1_2}(\mathscr{B}) \implies \mathbf{\Sigma^1_2}(\mathscr{K}_\sigma) \impliedby \mathbf{\Sigma^1_2}(\mathscr{R}).$$

The aim of this paper is to show that Con(ZFC) $\Rightarrow$ Con $\big(\mathsf{ZFC}+\mathsf{CH}+\mathbf{\Sigma^1_2}(\mathscr{D})+\neg\mathbf{\Sigma^1_2}(\mathscr{K}_\sigma)\big)$. In particular we get Con(ZFC) $\Rightarrow$ Con $\big(\mathsf{ZFC}+\mathsf{CH}+\mathbf{\Sigma^1_2}(\mathscr{D})+\neg\mathbf{\Sigma^1_2}(\mathscr{B})+\neg\mathbf{\Sigma^1_2}(\mathscr{R})\big)$

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1. Completely Doughnut Sets

In this section we introduce a pseudo topology on $[\omega]^\omega$, called the doughnut topology, which is related to the doughnut property and show that this pseudo topology has the same features as the Ellentuck topology, which was introduced by Erik Ellentuck in [El74] to prove that analytic sets are completely Ramsey. In our terminology, the non-empty basic open sets of the Ellentuck topology are the $(\emptyset, b)$-doughnuts, where $b \in [\omega]^\omega$. In fact, Ellentuck proved that a set $S \subseteq [\omega]^\omega$ is completely Ramsey if and only if it has the Baire property with respect to the Ellentuck topology, and that S is completely Ramsey null if and only if it is meager (or equivalently, nowhere dense) with respect to the Ellentuck topology. In the following we will see that a set is completely doughnut (defined below) if and only if it has the Baire property with respect to the doughnut topology and it is completely doughnut null if and only if it is meager (or equivalently, nowhere dense) with respect to the doughnut topology. Let us start by defining the doughnut topology:

For $a, b \subseteq \omega$ let

$$(a,b)^\omega := \big\{x \in [\omega]^\omega : a \subseteq x \subseteq b \wedge |x \setminus a| = |b \setminus x| = \omega\big\}.$$

The sets $(a,b)^\omega$ together with the sets $[\omega]^\omega$ and $\emptyset$ are the **basic open sets**. Since the intersection of two basic open sets might contain just one element, the basic open sets do not form a basis for a topology on $[\omega]^\omega$. However, since we use in the following just unions of basic open sets and the intersection of unions of basic open dense sets, let us say that a set S is open with respect to the *doughnut topology* if S is the union of basic open sets, and similarly we define nowhere dense and meager with respect to the doughnut topology.

For $a, b \subseteq \omega$ let

$$[a,b]^\omega := \big\{x \in [\omega]^\omega : a \subseteq x \subseteq b \wedge \big(|x \setminus a| = \omega \vee |b \setminus x| = \omega\big)\big\}.$$

Thus, a set $[a,b]^\omega$ is either an (a,b)-doughnut or empty.

In our terminology, the non-empty basic open sets of the Ellentuck topology are the doughnuts $[\emptyset, b]^\omega$, where $b \in [\omega]^\omega$.

FACT 1.1. For every doughnut $[a,b]^\omega$ there are $a',b' \in [a,b]^\omega$ such that $\emptyset \neq (a',b')^\omega \subseteq [a',b']^\omega \subseteq (a,b)^\omega \subseteq [a,b]^\omega$.

Proof. Take $a',b' \in [a,b]^\omega$ such that $a \subseteq a' \subseteq b' \subseteq b$ and each of the sets $a' \setminus a$, $b' \setminus a'$ and $b \setminus b'$ is infinite. $\dashv$

A set $S \subseteq [\omega]^\omega$ is **completely doughnut**, denoted by $\mathscr{D}_c$, if for each doughnut $[a,b]^\omega$ there is a doughnut $[a',b']^\omega \subseteq [a,b]^\omega$ such that $[a',b']^\omega \subseteq S$ *or* $[a',b']^\omega \cap S = \emptyset$. If we are always in the latter case, then S is called **completely doughnut null**.

Obviously, the complement of a completely doughnut set is also completely doughnut (*cf.* [El74, Lemma 5]). Moreover, every open set (w.r.t. the doughnut topology) is completely doughnut (*cf.* [El74, Lemma 4]):

FACT 1.2. If $S \subseteq [\omega]^\omega$ is an open set with respect to the doughnut topology, then S is completely doughnut.

Proof. If $S \subseteq [\omega]^\omega$ is open (w.r.t. the doughnut topology), then S is the union of basic open sets. Thus, for any doughnut $[a,b]^\omega$, $S \cap (a,b)^\omega$ is open. Hence, $S \cap (a,b)^\omega$ is either empty or contains a basic open set. Thus, by Fact 1.1, $S \cap [a,b]^\omega$ contains or is disjoint from some doughnut, and since $[a,b]^\omega$ was arbitrary, this implies that S is completely doughnut. $\dashv$

The following fact characterizes completely doughnut null sets in terms of nowhere dense sets (*cf.* [El74, Lemma 6]).

FACT 1.3. A set $S \subseteq [\omega]^\omega$ is completely doughnut null if and only if it is nowhere dense with respect to the doughnut topology.

Proof. Take any set $S \subseteq [\omega]^\omega$. By definition, S is completely doughnut null if for each doughnut $[a,b]^\omega$ there is a doughnut $[a',b']^\omega \subseteq [a,b]^\omega \setminus S$, and hence, by Fact 1.1, $[\omega]^\omega \setminus S$ contains an open dense set (w.r.t. the doughnut topology).

On the other hand, if $[\omega]^\omega \setminus S$ contains an open dense set (w.r.t. the doughnut topology), then for each doughnut $[a,b]^\omega$ there is a basic open set $(a',b')^\omega \subseteq (a,b)^\omega$ such that $(a',b')^\omega \cap S = \emptyset$. Now, for any doughnut $[a'',b'']^\omega \subseteq (a',b')^\omega$ we have $[a'',b'']^\omega \cap S = \emptyset$, and since $[a,b]^\omega$ was arbitrary, this implies that S is completely doughnut null. $\dashv$

Before we proceed, let us first define completely doughnut sets in terms of trees.

Let $\{0,1\}^{<\omega}$ be the set of all finite sequences of 0's and 1's. For $s,t \in \{0,1\}^{<\omega}$ we write $s \prec t$ if s is a proper initial segment of t, and we write $s \preccurlyeq t$ if $s \prec t$ or $s = t$. A set $T \subseteq \{0,1\}^{<\omega}$ is called a **tree**, if $s \in T$ and $t \prec s$ implies $t \in T$. If $T \subseteq \{0,1\}^{<\omega}$ is a tree and $s \in T$, then $T_s = \{t \in T : s \preccurlyeq t \vee t \prec s\}$. If $\langle s_0,\dots,s_n\rangle \in \{0,1\}^{<\omega}$ and $u \in \{0,1\}$, then $s^\frown u := \langle s_0,\dots,s_n,u\rangle$. A tree $T \subseteq \{0,1\}^{<\omega}$ is called **uniform**, if for all $s,t \in T$ of the same length we have

$$s^\frown 0 \in T \iff t^\frown 0 \in T \quad \textit{and} \quad s^\frown 1 \in T \iff t^\frown 1 \in T\,.$$

If T is a tree, then a set $\xi \subseteq T$ is called a **branch** through T, if for any $s,t \in \xi$ we have $s \preccurlyeq t$ or $t \prec s$ and ξ is maximal with respect to this property. The set of all branches through T is denoted by $[T]$. Notice that all branches through a finite uniform tree are of the same length. A tree $T \subseteq \{0,1\}^{<\omega}$ is called **perfect**, if for every $s \in T$ there is a $t \in T$ with $s \preccurlyeq t$ such that both $t^\frown 0$ and $t^\frown 1$ belong to T; such a sequence t is called a **splitting node** of T. The set of all splitting nodes of T is denoted by $\mathit{split}(T)$ and for $n \in \omega$, $\mathit{split}_n(T) := \big\{s \in \mathit{split}(T) : |\{t \in \mathit{split}(T) : t \prec s\}| = n\big\}$. Finally, the set $\mathit{splev}(T) := \{|t| : t \in \mathit{split}(T)\}$ denotes the set of all **split levels** of T.

Let ${}^\omega 2$ be the set of all functions from ω to $\{0,1\}$. For a set $a \subseteq \omega$, let $\chi_a \in {}^\omega 2$ be such that $\chi_a(n) = 1$ iff $n \in a$, and for $\xi \in {}^\omega 2$ let $\chi^{-1}(\xi) = \{n \in \omega : \xi(n) = 1\}$.

Fact 1.4. Each uniform perfect tree $T \subseteq \{0,1\}^{<\omega}$ corresponds in a unique way to a doughnut, and vice versa.

Proof. Let $T \subseteq \{0,1\}^{<\omega}$ be a uniform perfect tree. Let $D = \{\chi^{-1}(\xi) : \xi \in [T] \wedge \chi^{-1}(\xi) \in [\omega]^\omega\}$, then D is equal to the doughnut $[a,b]^\omega$, where $a = \bigcap_{\xi \in [T]}\{n \in \omega : \xi(n) = 1\}$ and $b \setminus a = \mathit{splev}(T)$.

On the other hand, if $a \subseteq b \subseteq \omega$ with $b \setminus a \in [\omega]^\omega$, then let $T \subseteq \{0,1\}^{<\omega}$ be the tree with $[T] = \{\xi \in {}^\omega 2 : a \subseteq \chi^{-1}(\xi) \subseteq b\}$. It is easy to see that T is a uniform perfect tree. ⊣

If T is a uniform perfect tree, then let $\mathit{donut}(T)$ denote the doughnut which—by Fact 1.4—corresponds to T.

The following lemma is similar to [El74, Lemma 7].

Lemma 1.5. With respect to the doughnut topology, a set $S \subseteq [\omega]^\omega$ is meager if and only if it is nowhere dense.

Proof. Let $S \subseteq [\omega]^\omega$ be any meager set. By Fact 1.3 it is enough to show that S is completely doughnut null. Since S is meager, there are countably many nowhere dense sets W_n such that $S = \bigcup_{n\in\omega} W_n$. For each $n \in \omega$, let $\mathcal{O}_n$ be an open dense set with $\mathcal{O}_n \cap W_n = \emptyset$. Let $[a,b]^\omega$ be any doughnut and let T^0 be the corresponding uniform perfect tree. Assume we already have constructed a uniform perfect tree T^n for some $n \in \omega$. For each $t \in \mathit{split}_n(T^n)$ consider the trees $T^n_{t^\frown 0}$ and $T^n_{t^\frown 1}$. By a successive amalgamation we can construct a uniform perfect tree T^{n+1} such that $\mathit{split}_n(T^{n+1}) = \mathit{split}_n(T^n)$, $[T^{n+1}] \subseteq [T^n]$ and $\mathit{donut}(T^{n+1}) \subseteq \mathcal{O}_n$. Finally, let $T = \bigcap_{n\in\omega} T^n$, which is by construction a uniform perfect tree. Now, $\mathit{donut}(T) = [a',b']^\omega$ for some $a' \subseteq b' \subseteq \omega$ with $b' \setminus a' \in [\omega]^\omega$, and by construction we have $[a',b']^\omega \subseteq [a,b]^\omega$ and $[a',b']^\omega \subseteq \bigcap_{n\in\omega}\mathcal{O}_n$, and since $S \cap \bigcap_{n\in\omega}\mathcal{O}_n = \emptyset$ we get $[a',b']^\omega \cap S = \emptyset$. ⊣

The following is a consequence of the preceding observations (*cf.* [El74, Theorem 9]).

Proposition 1.6. A set $S \subseteq [\omega]^\omega$ is completely doughnut if and only if S has the Baire property with respect to the doughnut topology.

Proof. Let $S \subseteq [\omega]^\omega$ be completely doughnut and let $\mathcal{O} = \bigcup\big\{(a,b)^\omega : [a,b]^\omega \subseteq S\big\}$. The set $\mathcal{O}$, as a union of open sets, is open. Since S is completely doughnut, for every doughnut $[a,b]^\omega$ there is a doughnut $[a',b']^\omega \subseteq [a,b]^\omega$, such that either $[a',b']^\omega \subseteq S$ or $[a',b']^\omega \cap S = \emptyset$. Hence, for every non-empty basic open set $(a,b)^\omega$ there is a non-empty basic open set $(a',b')^\omega \subseteq (a,b)^\omega$ such that $(a',b')^\omega \cap S \setminus \mathcal{O} = \emptyset$, which implies that the set $S \setminus \mathcal{O}$ is nowhere dense. Thus, S is the union of an open set and a nowhere dense set, and therefore has the Baire property.

On the other hand, let us assume that $S \subseteq [\omega]^\omega$ has the Baire property. Thus, there is an open set $\mathcal{O}$ such that $S \triangle \mathcal{O}$ is meager. Since meager sets are nowhere dense (by Lemma 1.5) and nowhere dense sets are completely doughnut null (by Fact 1.3), and since open sets are completely doughnut (by Fact 1.2), S is completely doughnut. $\dashv$

Remark. The set of completely doughnut null sets forms an ideal on $[\omega]^\omega$. This ideal, denoted by v^0, is related to Silver forcing and was studied by Jörg Brendle in [Br95]. For example he shows that $cov(v^0) \leq \mathfrak{r}$ (where $\mathfrak{r}$ denotes the *reaping number*) and that $\omega_1 = cov(v^0) < cov(\mathfrak{P}_2) = \omega_2 = \mathfrak{c}$ is consistent with ZFC (where $\mathfrak{P}_2$ denotes the Mycielski ideal).

2. Making Doughnuts of Perfect Sets of Cohen Reals

In the following we show how to make doughnuts of Cohen reals. First we consider the case when one Cohen real is added and then we show that a finite support iteration of length ω_1 of Cohen forcing, starting from $\mathbf{L}$, yields a model in which every $\boldsymbol{\Sigma}^1_2$-set is completely doughnut.

Let $\mathbb{C} := \langle \mathbf{C}, \geq \rangle$ be the Cohen partial ordering, where $\mathbf{C} := \{0,1\}^{<\omega}$ and for $s, t \in \mathbf{C}$ we stipulate $s \geq t$ iff $t \preccurlyeq s$. Remember that the algebra determined by $\mathbb{C}$ is the unique atomless complete Boolean algebra which has a countable dense subset (*cf.* [BJ95, Theorem 3.3.1]).

To define the forcing notion $\mathbb{P}$, which will be used later, we have to give some definitions.

Let $T \subseteq \{0,1\}^{<\omega}$ be a finite uniform tree. As mentioned in Section 1, all branches through a finite uniform tree are of the same length. For a finite uniform tree T, let $ht(T)$ denote the length of a branch through T. Finally, for a tree $T \subseteq \{0,1\}^{<\omega}$ let $T|_n = \{t \in T : |t| \leq n\}$.

Let $\mathbb{P} := \langle \mathbf{P}, \geq \rangle$ be the partial ordering, where

$$\mathbf{P} = \left\{T \subseteq \{0,1\}^{<\omega} : T \text{ is a finite uniform tree}\right\},$$

and for $T_1, T_2 \in \mathbf{P}$ we stipulate

$$T_1 \geq T_2 \iff ht(T_1) \geq ht(T_2) \quad \textit{and} \quad T_1|_{ht(T_2)} = T_2 \,.$$

Since $|\mathbf{P}| = \aleph_0$, the algebra determined by $\mathbb{P}$ is isomorphic to the algebra determined by $\mathbb{C}$, hence, the forcing notion $\mathbb{P}$ is isomorphic to $\mathbb{C}$. If G is a $\mathbb{P}$-generic filter over some model $\mathbf{V}$, then $T_G := \bigcup G$ is a uniform perfect tree. Moreover, by genericity, every $\xi \in [T_G]$ is Cohen generic over $\mathbf{V}$ (*cf.* [BJ95, Lemma 3.3.2]).

In the sequel we will not distinguish between the sets $[\omega]^\omega$ and ${}^\omega 2$, and all topological terms will refer to the usual topology on $[\omega]^\omega$ and ${}^\omega 2$, respectively.

Let $T \subseteq \{0,1\}^{<\omega}$ be any uniform perfect tree and let

$$\begin{array}{rccl} \sigma_T : & \omega & \longrightarrow & splev(T) \\ & n & \longmapsto & |t| \text{ for some } t \in split_n(T) \end{array}$$

be the function which enumerates the split levels of T. Further, we define a bijection Θ_T between $[T]$ and ${}^\omega 2$, by stipulating $\Theta_T(\xi)(n) := \xi\big(\sigma_T(n)\big)$ for $\xi \in [T]$. For any set $S \subseteq {}^\omega 2$, let $\Theta_T(S) := \{\Theta_T(\xi) : \xi \in S \cap [T]\}$. It is easy to see that if $S \subseteq {}^\omega 2$ is a $\boldsymbol{\Sigma}^1_2$-set, then also $\Theta_T(S)$ is a $\boldsymbol{\Sigma}^1_2$-set.

Now we are prepared to prove the following:

LEMMA 2.1. Let G be $\mathbb{P}$-generic over $\mathbf{V}$ and let $a, b \in \mathbf{V}$ be such that $a \subseteq b \subseteq \omega$ and $b \setminus a \in [\omega]^\omega$. Then for every $\mathbf{\Sigma^1_2}$-set $S \in V[G]$ with parameters in $\mathbf{V}$ there is a doughnut $[a', b']^\omega \subseteq [a, b]^\omega$ with $a', b' \in V[G]$, such that either $[a', b']^\omega \subseteq S$ or $[a', b']^\omega \cap S = \emptyset$.

Proof. Take any doughnut $[a, b]^\omega$ with a and b in $\mathbf{V}$ and let T be the uniform perfect tree which—by Fact 1.4—corresponds to $[a, b]^\omega$. Further, take any $\mathbf{\Sigma^1_2}$-set $S \subseteq [\omega]^\omega$ with parameters in $\mathbf{V}$. Put $S' = \Theta_T(S)$, then, since S is a $\mathbf{\Sigma^1_2}$-set, $S' = \bigcup_{\iota \in \omega_1} B_\iota$, where for each $\iota \in \omega_1$, B_ι is a Borel set (*cf.* [Je78, Theorem 95, p. 520]) with Borel code in $\mathbf{V}$. The following two cases are possible:

Case 1: There is an $\alpha \in \omega_1$ such that B_α is non-meager.

Case 2: For each $\iota \in \omega_1$, B_ι is meager.

We first consider case 1: So, let us assume that B_α is non-meager for some $\alpha \in \omega_1$. Since every Borel set has the Baire property, there is a non-empty basic open set $\mathcal{O}$ such that $\mathcal{O} \setminus B_\alpha$ is meager. Thus, there are countably many open dense sets D_n, $n \in \omega$, such that $\mathcal{O} \cap \bigcap_{n \in \omega} D_n \subseteq B_\alpha$. Following the proof of Lemma 1.5 (see also the proof of [DH00, Proposition 2.2]), by a successive amalgamation we can grow a uniform perfect tree $\tilde{T}$ such that $[\tilde{T}] \subseteq B_\alpha$. Let $T' \subseteq \{0,1\}^{<\omega}$ be the tree with $[T'] = \Theta_T^{-1}\big([\tilde{T}]\big)$, then T' is a uniform perfect subtree of T. Let $[a', b']^\omega = \mathit{donut}(T')$, then $[a', b']^\omega \subseteq [a, b]^\omega$ and $[a', b']^\omega \subseteq \Theta_T^{-1}(B_\alpha) \subseteq \Theta_T^{-1}(S') \subseteq S$. So far, we just worked in $\mathbf{V}$, but since the notions like *meager* and *nowhere dense* are absolute for Borel codes (*cf.* [Je78, Lemma 42.4]), $[a', b']^\omega \subseteq S$ is also valid in $\mathbf{V}[G]$, which completes case 1.

Let us now consider case 2: So, assume that for every $\iota \in \omega_1$, B_ι is meager. By genericity, $T_G = \bigcup G$ is a uniform perfect tree where $[T_G]$ avoids every meager set with Borel code in $\mathbf{V}$. Hence, $[T_G] \cap \bigcup_{\iota \in \omega_1} B_\iota = \emptyset$, which implies that $\Theta_T^{-1}\big([T_G]\big) \cap S = \emptyset$. Let $T' \subseteq \{0,1\}^{<\omega}$ be the tree such that $[T'] = \Theta_T^{-1}\big([T_G]\big)$, then T' is a uniform perfect subtree of T. Let $[a', b']^\omega = \mathit{donut}(T')$, then $[a', b']^\omega \subseteq [a, b]^\omega$ and $[a', b']^\omega \cap S = \emptyset$, which completes case 2 and the proof as well. $\dashv$

LEMMA 2.2. Let $\mathbb{C}_{\omega_1}$ be the finite support iteration of length ω_1 of Cohen forcing, starting from $\mathbf{L}$, and let G_{ω_1} be $\mathbb{C}_{\omega_1}$-generic over $\mathbf{L}$. Then $\mathbf{L}[G_{\omega_1}] \models \mathbf{\Sigma^1_2}(\mathscr{D}_c)$.

Proof. Let $\langle c_\iota : \iota < \omega_1 \rangle$ be the generic sequence of Cohen reals which corresponds to G_{ω_1}. Let $S \subseteq [\omega]^\omega$ be any $\mathbf{\Sigma^1_2}$-set with parameter r and let $[a, b]^\omega$ be any doughnut. By [Ku83, Chapter VIII, Lemma 5.14], there is a $\lambda \in \omega_1$ such that $a, b, r \in \mathbf{V}[G_\lambda]$, where $G_\lambda = \langle c_\iota : \iota < \lambda \rangle$. Since the forcing notions $\mathbb{P}$ and $\mathbb{C}$ are isomorphic, by Lemma 2.1 there is a doughnut $[a', b']^\omega$ in $\mathbf{V}[G_\lambda][c_\lambda]$ such that $[a', b']^\omega \subseteq [a, b]^\omega$ and either $\mathbf{V}[G_\lambda][c_\lambda] \models [a', b']^\omega \subseteq S$ or $\mathbf{V}[G_\lambda][c_\lambda] \models [a', b']^\omega \cap S = \emptyset$.

If we are in the former case (which corresponds to case 1 in the proof of Lemma 2.1), then $[a', b']^\omega \subseteq B'$ for some Borel set B' contained in S with Borel code in $\mathbf{V}[G_\lambda]$ and by absoluteness we get $\mathbf{V}[G_{\omega_1}] \models [a', b']^\omega \subseteq S$.

On the other hand, $\forall x (x \in [a', b']^\omega \to x \notin S)$ is a $\mathbf{\Pi^1_2}$-sentence with parameters in $\mathbf{V}[G_\lambda][c_\lambda]$ which holds in $\mathbf{V}[G_\lambda][c_\lambda]$, thus, by Shoenfield's Absoluteness Theorem (*cf.* [Je78, Theorem 98, p. 530]) we get $\mathbf{V}[G_{\omega_1}] \models [a', b']^\omega \cap S = \emptyset$.

Since the $\mathbf{\Sigma^1_2}$-set S and the doughnut $[a, b]^\omega$ were arbitrary, this completes the proof. $\dashv$

3. Conclusion

Before we can prove the main result of this paper, we have to give some definitions.

Let ${}^{\omega}\omega$ be the set of all functions from ω to ω. For $f, g \in {}^{\omega}\omega$ we write $f \leq^* g$, if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A family $\mathcal{F} \subseteq {}^{\omega}\omega$ is called **bounded**, if there is a function $g \in {}^{\omega}\omega$ such that for every $f \in \mathcal{F}$ we have $f \leq^* g$; otherwise, we call $\mathcal{F}$ **unbounded**.

Using the techniques given in [BJ95, Chapter 6, Section 5] (see also [Go93, Section 8]), one can show the following:

Lemma 3.1. *A finite support iteration of Cohen forcing preserves unbounded families. In particular, a finite support iteration of length ω_1 of Cohen forcing, starting from $\mathbf{L}$, yields a model in which ${}^{\omega}\omega \cap \mathbf{L}$ is unbounded.*

A tree $T \subseteq \omega^{<\omega}$ is called **superperfect**, if for every $s \in T$ there is a $t \in T$ such that $s \prec t$ and $\{n \in \omega : t^\frown n \in T\}$ is infinite. A set $F \subseteq {}^{\omega}\omega$ is $\boldsymbol{K_\sigma}$**-regular**, denoted by $\mathscr{K}_\sigma$, if F is either bounded or there is a superperfect tree T such that $[T] \subseteq F$.

Concerning $\boldsymbol{\Sigma}^1_2(\mathscr{K}_\sigma)$ we have the following two lemmata:

Lemma 3.2 ([Ju88, Theorem 1.1]).

$$\mathbf{V} \models \boldsymbol{\Sigma}^1_2(\mathscr{K}_\sigma) \iff \mathbf{V} \models \forall r \in {}^{\omega}\omega \, ({}^{\omega}\omega \cap \mathbf{L}[r] \text{ is bounded}) \,.$$

Lemma 3.3 ([Ju88, § 3]). *$\boldsymbol{\Sigma}^1_2(\mathscr{B}) \implies \boldsymbol{\Sigma}^1_2(\mathscr{K}_\sigma) \impliedby \boldsymbol{\Sigma}^1_2(\mathscr{R})$ are the only implications between the three properties $\mathscr{B}$, $\mathscr{R}$ and $\mathscr{K}_\sigma$.*

Now we are ready to prove the main result:

Theorem 3.4. *$\mathrm{Con}(\mathsf{ZFC}) \Rightarrow \mathrm{Con}\big(\mathsf{ZFC} + \mathsf{CH} + \boldsymbol{\Sigma}^1_2(\mathscr{D}_c) + \neg\boldsymbol{\Sigma}^1_2(\mathscr{K}_\sigma)\big)$. In particular, it is consistent with ZFC that $2^{\aleph_0} = \aleph_1$ and that there is a $\boldsymbol{\Sigma}^1_2$-set which is completely doughnut, but which has neither the Baire property nor the Ramsey property.*

Proof. Let $\mathbb{C}_{\omega_1}$ denote the finite support iteration of length ω_1 of Cohen forcing, and let G_{ω_1} be $\mathbb{C}_{\omega_1}$-generic over $\mathbf{L}$. Obviously we have $\mathbf{L}[G_{\omega_1}] \models 2^{\aleph_0} = \aleph_1$ and by Lemma 2.2 we also have $\mathbf{L}[G_{\omega_1}] \models \boldsymbol{\Sigma}^1_2(\mathscr{D}_c)$. On the other hand, by Lemma 3.1, $\mathbf{L}[G_{\omega_1}] \models {}^{\omega}\omega \cap \mathbf{L}$ *is unbounded*, and hence, by Lemma 3.2 we get $\mathbf{L}[G_{\omega_1}] \models \neg\boldsymbol{\Sigma}^1_2(\mathscr{K}_\sigma)$. In particular, by Lemma 3.3, there is a $\boldsymbol{\Sigma}^1_2$-set in $\mathbf{L}[G_{\omega_1}]$ which is completely doughnut, but which has neither the Baire property nor the Ramsey property. ⊣

Putting the previous results together, we get the following diagram:

$$\begin{tikzcd}
 & \boldsymbol{\Sigma}^1_2(\mathscr{K}_\sigma) & \\
\boldsymbol{\Sigma}^1_2(\mathscr{B}) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \boldsymbol{\Sigma}^1_2(\mathscr{R}) \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \\
 & \boldsymbol{\Sigma}^1_2(\mathscr{D}_c) \arrow[uu, Rightarrow, "/" marking] &
\end{tikzcd}$$

Considering this diagram, we get

QUESTION 1. $\mathbf{\Sigma^1_2}(\mathscr{K}_\sigma) \Longrightarrow \mathbf{\Sigma^1_2}(\mathscr{D}_c)$?

Haim Judah and Saharon Shelah have shown that $\mathbf{\Delta^1_2}(\mathscr{R}) \iff \mathbf{\Sigma^1_2}(\mathscr{R})$ (see [JS89, Theorem 2.10]). On the other hand, it is well-known that $\mathbf{\Delta^1_2}(\mathscr{B})$ does not imply $\mathbf{\Sigma^1_2}(\mathscr{B})$. This leads to

QUESTION 2. $\mathbf{\Delta^1_2}(\mathscr{D}_c) \iff \mathbf{\Sigma^1_2}(\mathscr{D}_c)$?

Further, they have also shown that $\mathbf{\Delta^1_2}(\mathscr{B}) \iff \forall r \in {}^\omega\omega \left(\mathit{Cohen}(\mathbf{L}[r]) \neq \emptyset\right)$, where $\mathit{Cohen}(\mathbf{L}[r])$ denotes the set of Cohen reals over $\mathbf{L}[r]$ (see [JS89, Theorem 3.1 (iii)]). Thus, in the model $\mathbf{L}[G_{\omega_1}]$ constructed above, every $\mathbf{\Delta^1_2}$-set has the Baire property, which motivates

QUESTION 3. $\mathbf{\Sigma^1_2}(\mathscr{D}_c) \Longrightarrow \mathbf{\Delta^1_2}(\mathscr{B})$?

According to [Br95], let r^0 and v^0 denote the ideals on $[\omega]^\omega$ of completely Ramsey null and completely doughnut null sets, respectively. Szymon Plewik has shown in [Pl86] that $\mathit{add}(r^0) = \mathit{cov}(r^0)$. Thus, we like to mention also

QUESTION 4. $\mathit{add}(v^0) = \mathit{cov}(v^0)$?

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