9. The Groups T, C, and D

In the sequel, T denotes the *tetrahedron-group*, C denotes the *cube-group* and D denotes the *dodecahedron-group*. Further, O denotes the *octahedron-group* and I denotes the *icosahedron-group*.

We already know that $O \cong C$ and $I \cong D$, so, we do not have to consider O and I.

Theorem 9.1. $T \cong A_4$, $C \cong S_4$ and $D \cong A_5$.

Proof. $T \cong A_4$: Let $1, 2, 3, 4$ denote the four faces of the tetrahedron, then each $\tau \in T$ can be considered as a permutation of $\{1, 2, 3, 4\}$ and the corresponding map $\varphi : T \to S_4$ is an injective homomorphism. Thus, T is isomorphic to a subgroup of S_4 of order $|T| = 12$. Further, each cycle $(i_1, i_2, i_3) \in S_4$ of length 3 can be realized by a rotation $\tau \in T$ of order 3. Thus, since A_4 is generated by the cycles of length 3, A_4 is isomorphic to a subgroup of T. Now, because $|A_4| = |T|$, this implies $T \cong A_4$.

$C \cong S_4$: Let $1, 2, 3, 4$ denote the four long diagonals of the cube, then each $\gamma \in C$ can be considered as a permutation of $\{1, 2, 3, 4\}$ and the corresponding map $\varphi : C \to S_4$ is an injective homomorphism (check that φ is injective). Thus, C is isomorphic to a subgroup of S_4 of order $|C| = 24 = |S_4|$, and therefore we get $C \cong S_4$.

$D \cong A_5$: Let $1, 2, 3, 4, 5$ denote the five different cubes we can put into a dodecahedron in such a way that each edge of each cube lies on one face of the dodecahedron. Thus, each $\delta \in D$ can be considered as a permutation of $\{1, 2, 3, 4, 5\}$ and the corresponding map $\varphi : D \to S_5$ is a homomorphism. Now, since a dodecahedron has 20 vertices, the five cubes have $5 \cdot 8 = 40$ vertices and there are $\binom{5}{2} = 10$ pairs of cubes, every two cubes have exactly two vertices in common and these two vertices are opposite each other. Now, if $\delta \in D$ is a rotation about an axis joining 2 opposite vertices through $2\pi/3$, then $\varphi(\delta)$ is a 3-cycle. On the other hand, for every 3-cycle $\sigma \in S_5$, there is a $\delta \in D$ such that $\varphi(\delta) = \sigma$. Hence, since by Proposition 7.14 every alternating group is generated by its 3-cycles, A_5 is isomorphic to a subgroup of D, and since $|A_5| = |D|$, we get $D \cong A_5$. ⊣

The subgroups of *T*. By Sylow's Theorem, T has 1 or 4 Sylow 3-subgroups which have order 3, and it has 1 or 3 Sylow 2-subgroups which have order 4. Further, T must also have a subgroup of order 2 (since by Cauchy's Theorem, a group of order 4 has always a subgroup of order 2), but we already know that T does not have a subgroup of order 6.

In the following we give a complete list of all subgroups of $A_4 \cong T$:

Of course, A_4 has exactly one subgroup of order 1, namely $\{\iota\}$, where ι is the identity, and it has exactly one subgroup of order 12, namely A_4 itself.

The subgroups of order 2 are: $\{\iota, (1,2)(3,4)\}$, $\{\iota, (1,3)(2,4)\}$, $\{\iota, (1,4)(2,3)\}$, and none of them is a normal subgroup of A_4.

There is just one subgroup of order 4, namely $\{\iota, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. Since a subgroup of order 4 is a Sylow 2-subgroup, by Corollary 8.11, $\{\iota, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is a normal subgroup of A_4, and further, it is isomorphic to $C_2 \times C_2$.

The 4 subgroups of order 3 are: $\{\iota, (1,2,3), (3,2,1)\}$, $\{\iota, (1,2,4), (4,2,1)\}$, $\{\iota, (1,3,4), (4,3,1)\}$ and $\{\iota, (2,3,4), (4,3,2)\}$. Since a subgroup of order 3 is a Sylow 3-subgroup, by Corollary 8.11, none of these subgroups of order 3 can be a normal subgroup of A_4.

Corollary 9.2. T is not simple.

Proof. Since T has a normal subgroup of order 4, T is not simple. ⊣

The subgroups of C of order 6, 8 and 12. The group C has 4 subgroups of order 3, namely rotations about a long diagonal through $2\pi/3$ and $-2\pi/3$. Each of these 4 Sylow 3-subgroups is isomorphic to C_3. Thus, C has 4 subgroups of order 6 (just turn the long diagonal), each of them is isomorphic to $D_3 \cong S_3$ and none of them is a normal subgroup of C. A subgroup of order 8 is a Sylow 2-subgroup, and since there are 3 subgroups of order 8, none of them is a normal subgroup. Further, each subgroup of order 8 is isomorphic to D_4. The group C has also a unique subgroup of order 12, which is isomorphic to T and since $|C : T| = 2$, this subgroup is a normal subgroup of C.

Corollary 9.3. C is not simple.

Proof. Since C has a normal subgroup of order 12, C is not simple. ⊣

The subgroups of D. A dodecahedron has 12 faces, 20 vertices and 30 edges. Remember that since $D \cong A_5$ and A_n is simple (for $n \geq 5$), D is simple, thus, D has no normal subgroups (except $\{\iota\}$ and D), in particular for $p = 2, 3, 5$, $|\operatorname{Syl}_p(D)| \neq 1$. In the following we give a complete list of all proper subgroups of D:

The subgroups of order 2 are the rotations about an axis joining midpoints of two opposite edges and since there are 30 edges, D has 15 subgroups of order 2.

A subgroup of order 3 is a Sylow 3-subgroup and therefore, $|\operatorname{Syl}_3(D)|$ is 4 or 10. Further, subgroups of order 3 are rotations about an axis joining opposite vertices and since there are 20 vertices, D has 10 subgroups of order 3.

A subgroup of order 4 is a Sylow 2-subgroup and therefore, $|\operatorname{Syl}_2(D)|$ is 3 or 5. Further, subgroups of order 4 are generated by rotations about three perpendicular axes joining midpoints of two opposite edges and since there are 30 edges, and each subgroup needs 6 edges, D has 5 subgroups of order 4 and each is isomorphic to $C_2 \times C_2$.

A subgroup of order 5 is a Sylow 5-subgroup and therefore, $|\operatorname{Syl}_5(D)|$ is 6. Indeed, subgroups of order 5 are rotations about an axis joining midpoints of opposite faces and since there are 12 faces, D has 6 subgroups of order 5.

It is not hard to see that D has 10 subgroups of order 6 and each is those subgroups is isomorphic to D_3.

Further, D has 6 subgroups of order 10 and each of those subgroups is isomorphic to D_5.

Finally we have 5 subgroups of order 12 and each of those subgroups is isomorphic to T.

Since D has no subgroups of order 15, 20 or 30, the 57 subgroups listed above are all proper subgroups of D.

Theorem 9.4. *D is simple.*

Proof. Let us define an equivalence relation "$\sim$" on D as follows:

$$a \sim b \iff \exists x \in D(xax^{-1} = b)$$

First we have to check that "$\sim$" is an equivalence relation:

$a \sim a$: $\iota a \iota^{-1} = a$.

$a \sim b \to b \sim a$: If $xax^{-1} = b$, then $x^{-1}bx = a$.

$a \sim b$ *and* $b \sim c \to a \sim c$: If $xax^{-1} = b$ and $yby^{-1} = c$, then $(yx)a(yx)^{-1} = c$.

The equivalence relation "$\sim$" induces a partition of D into five pairwise disjoint parts, namely

$$P_\iota = \{\iota\},$$
$$P_{2\pi/3} = \big\{ \textit{rotations through } 2\pi/3 \textit{ about axes joining opposite vertices} \big\},$$
$$P_\pi = \big\{ \textit{rotations through } \pi \textit{ about axes joining midpoints of opposite edges} \big\},$$
$$P_{2\pi/5} = \big\{ \textit{rotations through } 2\pi/5 \textit{ about axes joining centres of opposite faces} \big\},$$
$$P_{4\pi/5} = \big\{ \textit{rotations through } 4\pi/5 \textit{ about axes joining centres of opposite faces} \big\}.$$

We have $|P_\iota| = 1$, $|P_{2\pi/3}| = 20$, $|P_{2\pi}| = 15$, $|P_{2\pi/5}| = |P_{4\pi/5}| = 12$. Notice that $|D| = 60 = |P_\iota| + |P_{2\pi/3}| + |P_{2\pi}| + |P_{2\pi/5}| + |P_{4\pi/5}|$, thus, each element of D belongs to exactly one part of the partition.

Assume that $N \trianglelefteq D$ and let $a \in N$. Firstly, since N is a normal subgroup of D, N must contain all elements which are equivalent to a, which implies that N must be a union of some of the five parts. Secondly, since $N \leqslant D$, $|N|$ must divide $|D| = 60$. Now, since $P_\iota \subseteq N$, this is just possible if $N = P_\iota$ or $N = P_\iota \cup P_{2\pi/3} \cup P_{2\pi} \cup P_{2\pi/5} \cup P_{4\pi/5} = D$. Thus, $N = \{\iota\}$ or $N = D$, and therefore, D is simple. $\dashv$