5. Normal Subgroups

Before we define the notion of a normal subgroup, let us prove the following:

Fact 5.1. Let G be a group. If $H \leqslant G$ and $x \in G$, then

$$xHx^{-1} = \{xhx^{-1} : h \in H\}$$

is a subgroup of G.

Proof. Let xh_1x^{-1} and xh_2x^{-1} be in xHx^{-1}. Then $\big(xh_2x^{-1}\big)^{-1} = xh_2^{-1}x^{-1}$ and $\big(xh_1x^{-1}\big)\big(xh_2^{-1}x^{-1}\big) = x\big(h_1h_2^{-1}\big)x^{-1} \in xHx^{-1}$. So, by definition, $xHx^{-1} \leqslant G$. ⊣

This leads to the following definition.

Definition. Suppose that G is a group and that $N \leqslant G$, then N is called a **normal subgroup** of G if for all $x \in G$ we have

$$xNx^{-1} = N\,,$$

or equivalently, if for all $x \in G$, $xN = Nx$.

In particular, the trivial subgroups are normal and all subgroups of an abelian group are normal.

Notation. If $N \leqslant G$ ($N < G$) is a normal subgroup of G, then we write $N \trianglelefteq G$ ($N \triangleleft G$).

The following is just a consequence of Corollary 3.10:

Fact 5.2. If $H < G$ and $|G : H| = 2$, then $H \triangleleft G$.

Proof. By Corollary 3.10 we know that if $|G : H| = 2$, then for all $x \in G$ we have $xH = Hx$, and therefore $H \triangleleft G$. ⊣

Proposition 5.3. If $N \leqslant G$, then $N \trianglelefteq G$ if and only if for all $x \in G$ and all $n \in N$ we have

$$xnx^{-1} \in N\,.$$

Proof. If $N \trianglelefteq G$, then $xNx^{-1} = N$ (for all $x \in G$), thus, $xnx^{-1} \in N$ for all $x \in G$ and $n \in N$.
On the other hand, if $xnx^{-1} \in N$ for all $x \in G$ and $n \in N$, then $xNx^{-1} \subseteq N$ (for all $x \in G$). Further, replacing x by x^{-1} we get

$$N = x\,\underbrace{\big(x^{-1}Nx\big)}_{\subseteq N}\,x^{-1} \subseteq xNx^{-1}\,.$$

Hence, $xNx^{-1} = N$ (for all $x \in G$). ⊣

The following Fact is similar to Proposition 3.2:

Fact 5.4. If $K, H \trianglelefteq G$, then $(K \cap H) \trianglelefteq G$.

Proof. If $K, H \trianglelefteq G$, then, by Proposition 5.3, for all $x \in G$ and $n \in K \cap H$ we have $xnx^{-1} \in K$ (since $K \trianglelefteq G$) and $xnx^{-1} \in H$ (since $H \trianglelefteq G$), and therefore, $xnx^{-1} \in K \cap H$ (for all $x \in G$ and $n \in K \cap H$). ⊣

Notice that if $H \lhd K \lhd G$, then H is not necessarly a normal subgroup of G. To see this, let T be the tetrahedron-group, let ρ_1, ρ_2 and ρ_3 be the three elements of T of order 2, and let ι be the neutral element of T. Further, let $H = \{\iota, \rho_1\}$ and $K = \{\iota, \rho_1, \rho_2, \rho_3\}$. Since the group K is isomorphic to $C_2 \times C_2$, it is abelian and therefore we get $H \lhd K$. Further, for each $\tau \in T$ and $\rho \in K$, $\tau\rho\tau^{-1}$ has either order 1 or 2. Thus, $\tau\rho\tau^{-1} \in K$, which implies by Proposition 5.3 that $K \lhd T$. Finally, it is not hard to see that H is not a normal subgroup of T.

Let us now give some examples of normal subgroups:

(1) $T \lhd C$ (since $|C : T| = 2$).
(2) For $n \geq 3$, $C_n \lhd D_n$ (since $|D_n : C_n| = 2$).
(3) For $n \geq 1$, $\mathrm{SO}(n) \lhd \mathrm{O}(n)$ (since $|\mathrm{O}(n) : \mathrm{SO}(n)| = 2$).
(4) As we have seen above, T contains a normal subgroup which is isomorphic to $C_2 \times C_2$.
(5) For $n \geq 1$, $\mathrm{SL}(n) \lhd \mathrm{GL}(n)$: For all $B \in \mathrm{GL}(n)$ and $A \in \mathrm{SL}(n)$ we have $\det\left(BAB^{-1}\right) = \det(A) = 1$, thus, $BAB^{-1} \in \mathrm{SO}(n)$.

Definition. Suppose that G is a group. We define the **centre** $Z(G)$ of G by

$$Z(G) := \left\{a \in G : \forall x \in G(ax = xa)\right\}.$$

In other words, $Z(G)$ consists of those elements of G which commute with every element of G.

Fact 5.5. $Z(G) = G$ if and only if G is abelian.

Proof. If G is abelian, then for all $a \in G$ and for all $x \in G$ we have $ax = xa$, thus, $Z(G) = G$. On the other hand, $Z(G) = G$ implies that for all $a \in G$ and for all $x \in G$, $ax = xa$, thus, G is abelian. ⊣

Fact 5.6.

(a) $Z(G) \leqslant G$ (see Hw7.Q31.a).
(b) $Z(G) \trianglelefteq G$ (see Hw7.Q31.b).
(c) $Z(G)$ is abelian (see Hw7.Q31.c).
(d) If $H \leqslant Z(G)$, then $H \trianglelefteq G$ (see Hw7.Q31.d).

It is possible that the centre of a group is just the neutral element, *e.g.*, $Z(T) = \{\iota\}$.

Definition. Let G be a group and let H and K be subgroups of G. If $G = HK$, then we say that G is the **inner product** of H and K.

Proposition 5.7. Let G be a finite group and let $H, K \leqslant G$. Then

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.$$

Proof. First notice that $HK = \bigcup_{h \in H} hK$ and that $(H \cap K) \leqslant H$.

Now, for $h_1, h_2 \in H$ we have

$$h_1K = h_2K \iff h_1h_2^{-1} \in K,$$

and further we have

$$h_1(H \cap K) = h_2(H \cap K) \iff h_1h_2^{-1} \in (H \cap K) \iff h_1h_2^{-1} \in K.$$

Therefore,

$$|HK| = \Big|\bigcup_{h\in H} hK\Big| = \big|H : (H\cap K)\big| \cdot |K| = \frac{|H|}{|H\cap K|}\cdot |K| = \frac{|H|\cdot|K|}{|H\cap K|}\,.$$

⊣

Notice that if H and K are subgroups of a group G, then HK is not necessarly a subgroup of G (see Hw7.Q34). On the other hand, if at least one of these two subgroups is a normal subgroup, then HK is a subgroup of G:

Theorem 5.8. If $K \leqslant G$ and $N \trianglelefteq G$, then $KN = NK \leqslant G$.

Proof. Let us first show that $KN = NK$: Let $k \in K$ and $n \in N$, and let $n_1 = knk^{-1}$ and $n_2 = k^{-1}nk$. Then, since $N \trianglelefteq G$, $n_1, n_2 \in N$, and further we have

$$kn = n_1 k \ \ \textit{and} \ \ nk = kn_2\,,$$

which shows that $KN = NK$. To see that $KN \leqslant G$, pick two elements (k_1n_1) and (k_2n_2) of KN. We have to show that $(k_1n_1)(k_2n_2)^{-1} \in KN$:

$$(k_1n_1)(k_2n_2)^{-1} = k_1 \underbrace{n_1n_2^{-1}}_{=n_3\in N} k_2^{-1} = \underbrace{k_1k_2^{-1}}_{=k\in K} \underbrace{k_2n_3k_2^{-1}}_{=n\in N} = kn \in KN\,.$$

⊣

Let us give an example for Theorem 5.8: Consider the cube-group C. Let a, b, and c be the three axes joining centres of opposite faces and let $\rho_a, \rho_b, \rho_c \in C$ be the rotations about the axes a, b, and c respectively through π and let $\delta \in C$ be the rotation about the axis a through $\pi/2$. Now, let $N = \langle\{\rho_a, \rho_b, \rho_c\}\rangle$ and let $K = \langle\delta\rangle$. It is easy to see that K and N are both subgroups of C of order 4. Notice that $K \cong C_4$ and that $N \cong C_2 \times C_2$, so, K and N are not isomorphic, but they are both abelian. Let us now show that N is a normal subgroup of C: For this, we consider the set of axes $\{a, b, c\}$. Now, every $x \in C$ corresponds to a permutation τ_x on $\{a, b, c\}$, and $n \in N$ if and only if $\tau_n(a) = a$, $\tau_n(b) = b$, and $\tau_n(c) = c$, or in other words, $n \in N$ iff n corresponds to the identity permutation on $\{a, b, c\}$. For any $x \in C$ and $n \in N$, the permutation $\tau_{xnx^{-1}} = \tau_x\tau_n\tau_{x^{-1}}$ is the identity permutation on $\{a, b, c\}$, and hence, $xnx^{-1} \in N$, which shows that $N \lhd C$. Thus, by Theorem 5.8, $KN \leqslant C$. Since $|K \cap N| = 2$, by Proposition 5.7 we have $|KN| = \frac{|K|\cdot|N|}{|K\cap N|} = 8$ and it is not hard to see that $KN \cong D_4$.

Proposition 5.9. If K and H are subgroups of the finite group G, $|H \cap K| = 1$ and $|H| \cdot |K| = |G|$, then $HK = G = KH$.

Proof. Let us just prove that $HK = G$ (to show that $KH = G$ is similar). Since $HK = \{hk : h \in H \textit{ and } k \in K\} \subseteq G$, $HK = G$ if and only if $|HK| = |G|$, which implies that $h_1k_1 = h_2k_2$ if and only if $h_1 = h_2$ and $k_1 = k_2$. So, let us assume that $h_1k_1 = h_2k_2$, then $h_1^{-1}(h_1k_1)k_2^{-1} = h_1^{-1}(h_2k_2)k_2^{-1}$, and hence, $k_1k_2^{-1} = h_1^{-1}h_2 \in H \cap K$, but since $H \cap K = \{e\}$, this implies that $h_1 = h_2$ and $k_1 = k_2$. ⊣

The following proposition shows that if K and H are normal subgroups of G such that $|H \cap K| = 1$, then the elements of H commute with the elements of K and vice versa. Notice that this is stronger than just saying $KH = HK$.

PROPOSITION 5.10. If K and H are normal subgroups of G and $|H \cap K| = 1$, then for all $h \in H$ and all $k \in K$, $hk = kh$.

Proof. Let $h \in H$ and $k \in K$. Consider the element $hkh^{-1}k^{-1}$: On the one hand we have

$$\underbrace{h\overbrace{kh^{-1}k^{-1}}^{\in H}}_{\in H} \in H\,,$$

and on the other hand we have

$$\underbrace{\overbrace{hkh^{-1}}^{\in K}k^{-1}}_{\in K} \in K\,.$$

Thus, $hkh^{-1}k^{-1} \in H \cap K$, and since $|H \cap K| = 1$, $hkh^{-1}k^{-1} = e$, which implies $kh = hkh^{-1}k^{-1}(kh) = hk$. ⊣

PROPOSITION 5.11. If K and H are normal subgroups of G, then $KH \trianglelefteq G$.

Proof. For any $x \in G$, $xkhx^{-1} = \underbrace{(xkx^{-1})}_{\in K}\underbrace{(xhx^{-1})}_{\in H} \in KH$, thus, $xKHx^{-1} = KH$. ⊣

DEFINITION. A group G is called **simple** if it does not contain any non-trivial normal subgroup.

In particular, any abelian group which has a non-trivial subgroup cannot be simple, but there are also simple abelian groups, *e.g.*, the cyclic groups C_p, where p is prime (see Hw7.Q35). An example of a simple group which is not abelian is the dodecahedron-group D (as we will see later). On the other hand, there are many non-abelian groups which are not simple groups:

(1) The cube-group C, because $T \lhd C$.
(2) D_n for $n \geq 3$, because $C_n \lhd D_n$.
(3) $\mathrm{O}(n)$ for $n \geq 2$, because $\mathrm{SO}(n) \lhd \mathrm{O}(n)$.
(4) The tetrahedron-group T, because T contains a normal subgroup which is isomorphic to $C_2 \times C_2$.
(5) $\mathrm{GL}(n)$ for $n \geq 2$, because $\mathrm{SL}(n) \lhd \mathrm{GL}(n)$.