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Group Theory (Module 210PMA208)

Department of Pure Mathematics

Week 5

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21. (a) Is $C_3 \times C_4$ a cyclic group?

    (b) Does $C_3 \times C_4$ contain a cyclic group of order 6?

    (c) Let $H \leqslant C_3 \times C_4$. What are the possible values for $|H|$?

    (d) Is $C_3 \times C_2 \times C_2$ a cyclic group?

    (e) Does $C_3 \times C_2 \times C_2$ contain a cyclic group of order 6?

22. Let n be a positive integer and let $\mathrm{M}(n)$ be the set of all n by n matrices with real numbers as entries. Further, let $\mathrm{GL}(n) \subseteq \mathrm{M}(n)$ be the general linear group, $\mathrm{SL}(n) \subseteq \mathrm{M}(n)$ be the special linear group, $\mathrm{O}(n) \subseteq \mathrm{M}(n)$ be the orthogonal group, and $\mathrm{SO}(n) \subseteq \mathrm{M}(n)$ be the special orthogonal group.

    (a) Find a transversal for $\mathrm{SO}(n)$ in $\mathrm{O}(n)$ which is also a group and show that this group is isomorphic to C_2.

    (b) Find a transversal for $\mathrm{SL}(n)$ in $\mathrm{GL}(n)$ which is also a group and show that this group is isomorphic to $(\mathbb{R}^*, \cdot)$.

23. Let $\mathbb{I}^* = \{z \in \mathbb{C}^* : \mathrm{Re}(z) = 0\}$ and let $\mathbb{X} = \mathbb{R}^* \cup \mathbb{I}^*$.

    (a) Show that $\mathbb{X} \leqslant \mathbb{C}^*$.

    (b) Show that $\mathbb{U}/4 = \{e^{i\varphi} : 0 \leq \varphi < \pi/2\}$ is a transversal for $\mathbb{X}$ in $\mathbb{C}^*$.

    (c) Define an operation "$\circ$" on $\mathbb{U}/4$ such that $(\mathbb{U}/4, \circ)$ is a group.

24. Let T be the tetrahedron-group and let ρ_1, ρ_2 and ρ_3 be the three rotations through π about the axes joining midpoints of opposite edges.

    (a) Show that $\langle\{\rho_1, \rho_2, \rho_3\}\rangle$ is a subgroup of T of order 4.

    (b) Give the Cayley table for $\langle\{\rho_1, \rho_2, \rho_3\}\rangle$.

    (c) Show that $\langle\{\rho_1, \rho_2, \rho_3\}\rangle$ is isomorphic to $C_2 \times C_2$.

25. (a) What is the order of the group $(\mathbb{Z}_{15}, +)$?

    (b) Compute the order of each element of the group $(\mathbb{Z}_{15}, +)$.