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Group Theory (Module 210PMA208)

Department of Pure Mathematics

Week 4

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16. Show that every subgroup H of $(\mathbb{Z}, +)$ is of the form $H = m\mathbb{Z}$ (for some $m \in \mathbb{N}$).

17. The additive groups $4\mathbb{Z}$, $12\mathbb{Z}$ and $27\mathbb{Z}$ are subgroups of $(\mathbb{Z}, +)$, and so are the groups $4\mathbb{Z} \cap 12\mathbb{Z}$, $4\mathbb{Z} \cap 27\mathbb{Z}$ and $12\mathbb{Z} \cap 27\mathbb{Z}$. Thus, by Question 16, there are $m_1, m_2, m_3 \in \mathbb{N}$ such that

$$\begin{aligned} 4\mathbb{Z} \cap 12\mathbb{Z} &= m_1\mathbb{Z}, \\ 4\mathbb{Z} \cap 27\mathbb{Z} &= m_2\mathbb{Z}, \\ 12\mathbb{Z} \cap 27\mathbb{Z} &= m_3\mathbb{Z}. \end{aligned}$$

    (a) Compute m_1, m_2 and m_3.

    (b) Prove that for any $a, b \in \mathbb{N}$, $a\mathbb{Z} \cap b\mathbb{Z} = \operatorname{lcm}(a, b)\mathbb{Z}$, where $\operatorname{lcm}(a, b)$ is the least common multiple of a and b.

    (c) Show that $4\mathbb{Z} \cup 27\mathbb{Z}$ is not a subgroup of $(\mathbb{Z}, +)$.

18. (a) Evaluate $|\mathbb{Z} : 7\mathbb{Z}|$.

    (b) Find a transversal for $7\mathbb{Z}$ in $\mathbb{Z}$ which contains only even numbers.

19. Let C be the cube-group.

    (a) Describe and count the subgroups of C which are isomorphic to C_3, C_4, and $C_2 \times C_2$ respectively.

    (b) Does C have a proper non-abelian subgroup?

20. Let T be the tetrahedron-group. Let δ be the rotation through $2\pi/3$ about an axis joining a vertex to the centre of the opposite face and let ρ be the rotation through π about an axis joining midpoints of opposite edges.

    (a) Evaluate the order of δ and ρ, respectively.

    (b) Evaluate the order of the subgroup $\langle\delta\rangle \cap \langle\rho\rangle$.

    (c) Show that $\langle\{\delta, \rho\}\rangle = T$.