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Group Theory (Module 210PMA208)

Department of Pure Mathematics

Week 1

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1. Let $\mathbb{Q}^* := \mathbb{Q} \setminus \{0\}$, where $\mathbb{Q}$ denotes the set of all rational numbers.
   (a) Show that $(\mathbb{Q}^*, \cdot\,)$ has exactly one neutral element.
   (b) Show that in $(\mathbb{Q}^*, \cdot\,)$, every element has exactly one inverse.

2. Let "$\circ$" be a binary associative operation on the set S, so, for any $x, y, z \in S$ we have $x \circ (y \circ z) = (x \circ y) \circ z$.
   Show that for any $a, b, c, d \in S$ we have:
   $$(a \circ b) \circ (c \circ d) = \big(a \circ (b \circ c)\big) \circ d$$

3. Show that the binary operation
   $$\begin{array}{rccc} \sharp : & \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} \\ & (x, y) & \mapsto & x \cdot (y + 1) \end{array}$$
   is neither commutative nor associative.

4. Let $A = \{1, 2, 3, 4, 6, 12\}$ and let
   $$\begin{array}{rccc} \star : & A \times A & \to & A \\ & (a, b) & \mapsto & \gcd(a, b) \end{array}$$
   where $\gcd(a, b)$ denotes the greatest common divisor of a and b.
   (a) Show that the operation "$\star$" is commutative and associative.
   (b) Show that $(A, \star)$ has a neutral element.
   (c) Why is $(A, \star)$ not a group?

5. Let "$\bullet$" be the binary operation on $\mathbb{Q}^*$ defined as follows:
   $$\begin{array}{rccc} \bullet : & \mathbb{Q}^* \times \mathbb{Q}^* & \to & \mathbb{Q}^* \\ & (p, q) & \mapsto & p \cdot (q + q) \end{array}$$
   Show that $(\mathbb{Q}^*, \bullet)$ is an abelian group.