SISS: Examples of classical integrable systems

Abstract

I'll explain how to use Hamiltonian reduction to compute trajectories of the open Toda lattice. Then we'll move on to another example and look at the algebraic construction of the classical Gaudin spin chain we encountered in Qiao's, Kolya's and my previous lectures.

Some more on the Toda lattice. Consider the real Lie algebra $\mathfrak{g} = sl_{n+1}(\mathbf{R})$. It has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = so_{n+1}(\mathbf{R})$ is the Lie subalgebra of skew-symmetric matrices, and $\mathfrak{p}$ is the vector subspace of symmetric matrices in $\mathfrak{g}$. Let's also introduce the notation $\mathfrak{b}$ for the Lie subalgebra of upper triangular matrices, and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

We shall use the Killing form (normalized as the trace form of the defining representation of $\mathfrak{g}$) to identity $\mathfrak{g}^*$ with $\mathfrak{g}$. Note that under this identification, we have $\mathfrak{b}^\perp = \mathfrak{n}, \mathfrak{k}^\perp = \mathfrak{p}$. We also have the vector space decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}$. Hence the canonical identifications $\mathfrak{b}^* = \mathfrak{g}^*/\mathfrak{b}^\perp$, $\mathfrak{k}^* = \mathfrak{g}^*/\mathfrak{k}^\perp$ give us identifications $\mathfrak{b}^* \simeq \mathfrak{p}, \mathfrak{k}^* \simeq \mathfrak{n}$. These last identifications are non-canonical in the sense that they depend on our choice of $\mathfrak{p}$ as the direct summand of $\mathfrak{g}$ complementary to $\mathfrak{n}$. However, the decomposition

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n} \simeq \mathfrak{b}^* \oplus \mathfrak{k}^*$$

will prove very handy.

The phase space of the Toda lattice arises from considering the quotient Lie algebra $\mathfrak{s} = \mathfrak{b}/[\mathfrak{n}.\mathfrak{n}]$. Our model for its dual $\mathfrak{s}^*$ is the subspace $\mathfrak{p}_1$ of tridiagonal, symmetric traceless matrices. Once again, $\mathfrak{p}_1$ is set in duality with $\mathfrak{s}$ by the Killing form of $\mathfrak{g}$. Precomposing the quotient projection with the coadjoint action gives an action of B on $\mathfrak{s}^* \simeq \mathfrak{p}_1$. There is an open orbit

$$\mathcal{O} = \left\{ \sum_{k=1}^{n} p_k E_{k,k+1} + e^{q_k}(E_{k,k+1} + E_{k+1,k}) \;\middle|\; p_k, q_k \in \mathbf{R} \right\}$$

which is the phase space of the Toda lattice. It's homework to compute the Poisson brackets of the coordinates p_k, q_k.

The Toda Hamiltonian can be constructed by *symplectic induction.* Recall that the group $G = SL_{n+1}(\mathbf{R})$ has an Iwasawa decomposition $G = KB$ where $K = SO_{n+1}(\mathbf{R})$ and B is the subgroup of upper triangular matrices with positive entries on the diagonal. Another way to put this is that the group $K \times B$ acts freely and transitively on G by

$$(k, b) \cdot g = kgb^{-1}$$

Let us identify $T^*G = G \times \mathfrak{g}$ by left translations and the Killing form. This means that for $X \in \mathfrak{g}, Z \in \mathfrak{g}^*$, the covector (g, Z) returns the value $\langle Z, X \rangle$ on the tangent vector $L'_g(X)$. Then the action of $K \times B$ on G lifts to the action

$$(k, b) \cdot (g, Z) = (kgb^{-1}, \mathrm{Ad}_b Z)$$

which has equivariant moment map

$$\mu(g, Z) = (\mathrm{proj}_{\mathfrak{n}}(Z), \mathrm{proj}_{\mathfrak{p}}(\mathrm{Ad}_g Z))$$

Let us now consider the diagonal action of $K \times B$ on the twisted product symplectic manifold

$$\mathcal{M} = T^*G \times_- \mathcal{O}$$

where we let K act trivially on $\mathcal{O}$ and take the symplectic on the product as the difference of symplectic forms on the factors.

$$(k, b) \cdot (g, Z, X) = (kgb^{-1}, \mathrm{Ad}_b Z, \mathrm{Ad}_b X)$$

The moment map for this action is

$$\begin{aligned} \mu : T^*G \times_- \mathcal{O} &\to \mathfrak{n} \oplus \mathfrak{p} \\ (g, Z, X) &\longmapsto (\mathrm{proj}_{\mathfrak{n}}(Z), \mathrm{proj}_{\mathfrak{p}}(\mathrm{Ad}_g Z) - X) \end{aligned}$$

The group $K \times B$ acts freely on $\mathcal{M}$, and a set of orbit representatives is

$$\{(1, Z, X) \big| Z \in \mathfrak{g}, X \in \mathcal{O}\}$$

Hence the orbits that are contained in the zero level of the moment map are exactly those through

$$\{(1, X, X) \big| X \in \mathfrak{g}, X \in \mathcal{O}\}$$

$$(kb^{-1}, \mathrm{Ad}_b X, \mathrm{proj}_{\mathfrak{p}_1} \mathrm{Ad}_b X)$$

As a sanity check, these should be preserved by the action. So we have a symplectomorphism

$$\mathcal{O} \to \mu^{-1}(0)/K \times B$$

(since we can lift tangent vectors in the quotient to tangent vectors whose G-component is 0).

What does this complicated description of $\mathcal{O}$ buy us? Consider the quadratic casimir $\mathcal{H} \in \mathrm{Fun}(\mathfrak{g}^*)$. We can regard $\mathcal{H}$ as a function on T^*G which is invariant under both left and right translations. Under our identifications, $\mathcal{H}$ is just the Killing form:

$$\mathcal{H}(g, Z) = (Z, Z)$$

This descends to a function H on the quotient space $\mathcal{O} \simeq \mu^{-1}(0)/K \times B$, where it coincides with the Toda Hamiltonian

$$H(p_k, e^{q_k}) = (p, p) +$$

Explicit solution of Toda lattice. The virtue of this construction is that the Hamiltonian

$$\mathcal{H}(g, Y) = \frac{1}{2}\langle Y, Y\rangle$$

is really easy to integrate. Its Hamiltonian vector field is

$$v_{\mathcal{H}}(g, Y) = (L'_g(Y), 0)$$

whose trajectory through (g, Y) is $\gamma(t) = (ge^{tY}, Y)$. We can project these flows to find the trajectories of the Toda lattice. The trajectory through

$$X = (1, X, X) \longmapsto (e^{tX}, X, X) = (k(t)b^{-1}(t), X, X) = (1, \mathrm{Ad}_{b(t)}(X), \mathrm{Ad}_{b(t)}(X))$$

So if $e^{tX} = k(t)b^{-1}(t)$, we have

$$X(t) = \mathrm{proj}_{\mathfrak{p}_1} \mathrm{Ad}_{b(t)} X$$

Loop algebras and the Gaudin model. Now disregard all notations of previous sections. Let's briefly recall the construction of integrable systems with linear Poisson brackets. We take a Lie algebra $\mathfrak{g}$ with an invariant non-degenerate bilinear form, and two subalgebras $\mathfrak{g}_\pm$ such that we have a vector space decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Let $r = P_+ - P_-$ be the difference of the two projections with respect to this decomposition. By the general nonsense, r satisfies the mCBYE, and therefore Casimirs in $S(\mathfrak{g})^{\mathfrak{g}}$ remain commutative with respect to the r-bracket. Moreover, the Lie algebra $\mathfrak{g}_r$ has Lie bracket $[x, y]_r = [x_+, y_+] - [x_-, y_-]$ and hence splits as a *direct sum of Lie algebras* $\mathfrak{g}_r = \mathfrak{g}_+ \oplus \mathfrak{g}_-^{op}$. So we have a map of Lie algebras $\mathfrak{g} \to \mathfrak{g}_+$ which gives a Poisson map $\mathfrak{g}_+^* \hookrightarrow \mathfrak{g}^*$, so the projection $S(\mathfrak{g}) = S(\mathfrak{g}_+) \oplus \mathfrak{g}_- S(\mathfrak{g})$ is a Poison map. The upshot is we get a Poisson-commutative family on $\mathfrak{g}^*$. More generally, we could project onto any ideal in $\mathfrak{g}_r$.

Here's an example of this situation. Let $\mathfrak{g}$ be a complex simple Lie algebra. Consider a finite set of points

$$D = \{z_1, \ldots, z_N\} \subset \mathbf{C} \subset \mathbf{P}^1$$

and let $\mathfrak{g}(D)$ be the Lie algebra of rational functions valued in $\mathfrak{g}$ which vanish at infinity and are regular outside D. Note that we put no restriction on the order of poles at D. We also have the localizations

$$\mathfrak{g}_{z_i} = \mathfrak{g} \otimes \mathbf{C}((z - z_i))$$

and their direct sum

$$\mathfrak{g}_D = \bigoplus_{z_i \in D} \mathfrak{g}_{z_i}$$

We have an embedding

$$\mathfrak{g}(D) \to \mathfrak{g}_D$$

sending a rational function to its Laurent expansions at each point in D. Write $\mathfrak{g}_{z_i}^+ = \mathfrak{g} \otimes \mathbf{C}[[z - z_i]]$, $\mathfrak{g}_D^+ = \oplus_i \mathfrak{g}_{z_i}^+$. Then expanding a rational function into its principal parts gives us the vector space decomposition

$$\mathfrak{g}_D = \mathfrak{g}_D^+ \oplus \mathfrak{g}(D)$$

The Killing form gives us an invariant bilinear form on $\mathfrak{g}_D$

$$\langle X(z), Y(z)\rangle = \sum_i \mathrm{Res}_{z_i}\langle X_i, Y_i\rangle$$

which sets $\mathfrak{g}_D^+, \mathfrak{g}(D)$ into duality as isotropic subspaces. Hence we have the coadjoint action of $\mathfrak{g}_D^+$ on $\mathfrak{g}(D)$, and we have the rational functions with simple poles forming an invariant subspace $\mathfrak{g}(D)_1 \subset \mathfrak{g}(D)$. It is $\mathfrak{g}(D)_1$ that will be the phase space of the Gaudin model. Let $\mathfrak{g}_{z_i}^{++}$ be the formal series with zero constant term, and $\mathfrak{g}_D^{++}$ their direct sum. We have

$$\mathfrak{g}_D^+/\mathfrak{g}_D^{++} \simeq \mathfrak{g}^N = \oplus_i \mathfrak{g}$$

and $\mathfrak{g}^N$ is in duality with $\mathfrak{g}(D)_1$. We also have Poisson maps $S(\mathfrak{g}_D) \to S(\mathfrak{g}_D^+) \to S(\mathfrak{g}_D^+/\mathfrak{g}_D^{++})$, and this latter space we shall think of as polynomial functions on $\mathfrak{g}(D)_1$. Hence we get Poisson commuting functions on $\mathfrak{g}(D)_1$

Let $L(z) \in \mathfrak{g}(D)_1 \otimes \mathfrak{g}^N$ be the canonical element, choose a representation V of $\mathfrak{g}$, and extend it to an evaluation representation $V(z) = V \otimes \mathbf{C}(z)$. Then $L(z)$ can be regarded as a matrix whose elements are functions on $(\mathfrak{g}^N)^*$. The Lie-Poisson bracket can be written simply using the r-matrix

$$r_V(u,v) = \frac{t_V}{u-v}$$

where t_V is evaluation of the tensor Casimir in $V \otimes V$. Then we have

$$\{L(u) \otimes L(v)\} = [r_V(u,v), L(u) \otimes 1 + 1 \otimes L(v)]$$

Our Hamiltonians will come from projecting Casimir elements in $S(\mathfrak{g})$ onto $S(\mathfrak{g}^N)$. Actually, there are no Casimirs in $S(\mathfrak{g})$ and in order to find some it is necessary to pass to an appropriate completion. In the case $\mathfrak{g} = sl_2$, the simplest Casimir is

$$\mathcal{H} = \sum_{i=1}^N \sum_{k\in\mathbf{Z}} h_k^{(i)} h_{k-1}^{(i)} + e_{k-1}^{(i)} f_k^{(i)} + e_k^{(i)} f_{k-1}^{(i)}$$

We now must project this to $S(\mathfrak{g}^N)$ with respect to the decomposition

$$\mathfrak{g}_D = S(\mathfrak{g}^N) \oplus (\mathfrak{g}(D)S(\mathfrak{g}_D) + \mathfrak{g}_D^{++}S(\mathfrak{g}_D))$$

The result is

$$H = \frac{1}{2}\sum_{i=1}^N \sum_{k\geq 1} h_k^{(i)} h_{k-1}^{(i)} + 2e_{k-1}^{(i)} f_k^{(i)} + 2e_k^{(i)} f_{k-1}^{(i)}$$

Restricting to the Casimir elements gives a map of Poisson algebras, and under the natural pairing with $\mathfrak{g}(D)_1$ we recover the spectral invariants of $L(z)$.