Review problems.

Problem 1. Let M_3 be the vector space of 3×3 real matrices, equipped with the inner product

$$\langle A, B \rangle = \operatorname{trace}(AB)$$

(recall that the trace of a matrix is the sum of its diagonal entries.)

Let $\mathfrak{h} \subset M_3$ be the subset of diagonal matrices D with $\operatorname{trace}(D) = 0$. Show that $\mathfrak{h}$ forms a subspace of M_3, find the dimension of $\mathfrak{h}$, and find an orthonormal basis for $\mathfrak{h}$.

Problem 2. Let M_2 be the vector space of 2×2 real matrices, and let $\mathfrak{sl}_2$ be the subset of matrices of trace zero:

$$\mathfrak{sl}_2 = \left\{ A \in M_2 \;\middle|\; \operatorname{trace}(A) = 0 \right\}$$

If

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

then the set $\mathcal{B} = \{E, F, H\}$ is a basis for $\mathfrak{sl}_2$. Define a linear transformation $T : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ by

$$T(A) = EA - AE$$

Write the matrix of T with respect to the basis $\mathcal{B}$. Determine the rank of T, and find bases for its kernel and image.

Problem 3. Let $P_{\leq 3}$ be the vector space of polynomials with real coefficients of degree at most 3, and define an inner product on P by

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx$$

Find an orthogonal basis for P_3.

Problem 4. If

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

calculate $\exp(tA)$. Solve the initial value problem $x'(t) = Ax, \quad x(0) = c$ where

$$c = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Problem 5. Find a solution to the differential equation $x'(t) = Cx(t) + f(t)$, where

$$C = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$f(t) = \begin{bmatrix} e^{2t} \\ \sin t \\ t \end{bmatrix}$$

Problem 6. Consider the mass-springs system with equations of motion

$$\begin{aligned} x_1'' &= -4x_1 + 2x_2 \\ x_2'' &= 2x_1 - 5x_2 \end{aligned}$$

Solve the initial value problem with $x_1(0) = 0,\ x_2(0) = 2,\ x_1'(0) = 0 = x_2'(0)$.

Problem 7. Consider the function $f(x)$ defined for $0 < x < 1$ by

$$f(x) = \begin{cases} 1/2 & \text{if } 0 < x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

Compute the Fourier cosine series of the even 1-periodic extension of $f(x)$.

Problem 8. Find a solution $u(x,t)$ to the following initial-boundary value problem

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t < 0$$

$$u(0,t) = 0 = \frac{\partial u}{\partial x}(1,t)$$

$$u(x,0) = f(x)$$

where $f(x)$ is the function defined in Problem 8.

Problem 9. Let T be the linear transformation in Problem 2, and let H be the matrix defined in Problem 2. Solve the initial value problem

$$x'(t) = T(x), \quad x(0) = H$$