# LINEAR PROGRAMMING AND HAMILTON-JACOBI-BELLMAN EQUATIONS

## 1. Introduction

In linear programming, problems appear in pairs—a primal problem has its dual, which contains useful information about the primal problem, reviewed in the next section. We want to apply this technique to optimal control problems. But optimal control problems are in general non-linear. However, we can relax the problem—instead of looking for an optimal trajectory/control, we search for a measure $\mu$ to minimize the "generalized" cost functional, with constraints that generalize the restrictions in the original optimal control problem (dynamics, initial condition, etc.). The "relaxed" problem becomes linear and we can find its dual and then use the dual to study the primal problem.

## 2. Linear Programming

2.1. **Finite Dimensional Case.** Recall that in linear programming, we are given a vector $c \in \mathbb{R}^n, b \in \mathbb{R}^m$ and a matrix $A \in \mathbb{R}^{m\times n}$. The *primal problem* is to find $\hat{x} \in \mathbb{R}^n$ to

$$\text{(P)} \qquad \begin{cases} \text{minimize}\, c\cdot x \\ \text{s. t.} \quad Ax = b, x \geq 0. \end{cases}$$

and the corresponding *dual problem* is to find $\hat{y} \in \mathbb{R}^m$ to

$$\text{(D)} \qquad \begin{cases} \text{maximize}\, y\cdot b \\ \text{s. t.} \quad A^T y \leq c. \end{cases}$$

A vector $x \in \mathbb{R}^n$ is called *feasible* for (P) if $Ax = b, x \geq 0$; Similarly a vector $y \in \mathbb{R}^m$ is called *feasible* for *problem dual* if $A^T y \leq c$. The main theorem in finite dimensional linear programming is the following

**Theorem 2.1** (Strong Duality)**.** *Assume that the sets of feasible vectors for* (P) *and* (D) *are both nonempty. Then*

(1) *problem* (P) *has a solution* $\hat{x}$ *and problem* (D) *has a solution* $\hat{y}$.
(2) *We have*

$$c\cdot\hat{x} = \hat{y}\cdot b, \tag{1}$$

*or equivalently,*

$$\hat{x}\cdot(A^T\hat{y} - c) = 0. \tag{2}$$

The last equality gives *complementary slackness condition*:

$$(A^T\hat{y} - c)_i = 0 \text{ whenever } \hat{x}_i \neq 0. \tag{3}$$

2.2. **Infinite Dimensional Case.** An infinite dimensional analogue of the linear programming problem discussed above is this. Let $X, Y$ be real topological vector spaces, and assume $X$ is endowed with a partial ordering, which is compatible with the linear structure and which induces a partial ordering on its dual space $X^*$. We denote the pairing between a space and its dual by $\langle\cdot,\cdot\rangle$.

We are give $c^* \in X^*, b \in Y$ and a bounded linear operator $A : X \to Y$ whose adjoint is $A^* : Y^* \to X^*$. The primal problem is to find $\hat{x} \in X$ to

$$\begin{cases} \text{minimize}\langle c^*, x\rangle \\ \text{s. t.} \quad Ax = b, x \geq 0. \end{cases} \tag{P}$$

and the corresponding *dual problem* is to find $\hat{y}^* \in Y^*$ to

$$\begin{cases} \text{maximize}\langle y^*, b\rangle \\ \text{s. t.} \quad A^*y^* \leq c^*. \end{cases} \tag{D}$$

In the infinite dimensional case, we don't in general have strong duality as finite dimensional case. But we do have the *weak duality*:

**Theorem 2.2** (Weak Duality)**.**

(1) *If $x$ is feasible for* (P) *and $y^*$ is feasible for* (D)*, then*

$$\langle c^*, x\rangle \geq \langle y^*, b\rangle.$$

(2) *If for some feasible $\hat{x}$ and $\hat{y}^*$ satisfying*

$$\langle c^*, \hat{x}\rangle = \langle \hat{y}^*, b\rangle,$$

*then $\hat{x}$ is optimal for* (P) *and $\hat{y}^*$ is optimal for* (D)*.*

*The above identity is equivalent to the complementary slackness condition*

$$\langle A^*\hat{y}^* - c^*, \hat{x}\rangle = 0.$$

It's useful to define the *duality gap*

$$\Delta := \inf\{\langle c^*, x\rangle | Ax = b, x \geq 0\} - \sup\{\langle y^*, b\rangle | A^*y^* \leq c^*\}. \tag{4}$$

Clearly, by Theorem 2.2, $\Delta \geq 0$.

## 3. Applications to Dynamic Programming

3.1. **Ergodic Case.**

3.1.1. *Setup of the Problem.* The original problem is to find a measurable control function $\alpha(\cdot)$ (or progressively measurable process, in the stochastic case) with values in $V$ to minimize the *cost criterion*

$$Cost = \lim_{T\to\infty} \frac{1}{T} \int_0^T L(x(t), \alpha(t))dt, \tag{5}$$

where the state $x(t)$ is governed by some dynamics (deterministic or stochastic) of the form

$$\begin{cases} dx = f(t, x(t), \alpha(t))dt + \sigma(t, x(t), \alpha(t))dw \\ x(0) = x \end{cases} \tag{6}$$

where $w$ is $n-$dimensional Wiener process. We assume all functions are periodic in $x$. This minimization problem isn't linear. But if we relax the problem and search for instead a measure $\mu$ on $\Omega := \mathbb{T}^n \times V$ to

$$\begin{cases} \text{minimize} \displaystyle\int_\Omega L(x, v)d\mu \\ \text{s. t.} \quad \mu \geq 0, \mu(\Omega) = 1, \displaystyle\int_\Omega T(x, v)\phi d\mu = 0, \forall \phi = \phi(x) \end{cases} \tag{P1}$$

where $T(x, v) : C^\infty(\mathbb{T}^n) \to C^\infty(\Omega)$ is an operator with parameters $x \in \mathbb{T}^n, v \in V$. The last condition, $\mu$ being flow invariant, generalizes the classical requirement that the cost is computed along a curve.

*Remark* 3.1. Of course, the operator $T$ depends on the dynamics (6) if we want the generalized problem (P1) to be related to the original problem. For instance [5], if the dynamics are

$$dx = \alpha(t)dt + \sigma dw,$$

$\sigma > 0$ is deterministic, then $T(x, v)$ should be the *infinitesimal generator* of the controlled diffusion [4]

$$T(x, v)\phi = v \cdot D\phi(x) + \frac{\sigma^2}{2}\Delta\phi.$$

3.1.2. *Discrete Approximation and Dual.* We discretize the problem and do some formal calculation and then get the dual problem. We want to use the dual problem of the discrete primal problem to guess the dual problem of (P1).

Take points $x_i \in \mathbb{T}^n (1 \leq i \leq M)$ and $v_j (1 \leq j \leq N) \in V$, and assume $\mu$ is accumulated at points $(x_i, v_j)$ with $\mu(x_i, v_j) = \mu_{ij}$. Denote $\lambda_{ij} = L(x_i, v_j)$ and $T^*_{ij}$ is the formal adjoint of $T(x_i, v_j)$. Then the *discrete primal problem* is

$$\begin{cases} \text{minimize} \displaystyle\sum_{i,j} \lambda_{ij}\mu_{ij} \\ \text{s. t.} \quad \displaystyle\sum_j T^*_{ij}\mu_{ij} = 0, \forall i, \sum_{i,j} \mu_{ij} = 1, \mu_{ij} \geq 0 \end{cases} \tag{P1'}$$

Then the above problem is of form (P) with

$$x = (\mu_{11}, \cdots, \mu_{1N}, \cdots, \mu_{M1}, \cdots, \mu_{MN})^T$$
$$c = (\lambda_{11}, \cdots, \lambda_{1N}, \cdots, \lambda_{M1}, \cdots, \lambda_{MN})^T$$
$$b = (0, \cdots, 0, 1)^T \in \mathbb{R}^{M+1}$$

and an $(M+1) \times (MN)$ matrix

$$A = \begin{pmatrix} T^*_{11} & \cdots & T^*_{1N} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & T^*_{21} & \cdots & T^*_{2N} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & T^*_{M1} & \cdots & T^*_{MN} \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$

Accordingly, the discrete dual problem is

$$\text{(D1')} \qquad \begin{cases} \text{maximize}\, \beta \\ \text{s. t.} \quad T_{ij} y_i + \beta \le \lambda_{ij}, \forall 1 \le i \le M, 1 \le j \le N. \end{cases}$$

where the unknown is $y = (y_1, \cdots, y_M, \beta)$.

Based on the discrete dual problem, we *guess* the dual problem of (P1) is

$$\text{(D1)} \qquad \begin{cases} \text{maximize}\, \beta \\ \text{s. t.} \quad T(x,v)u(x) + \beta \le L(x,v), \forall (x,v) \in \Omega, \end{cases}$$

where the unknown is $y = (u(x), \beta)$.

3.1.3. *Interpretations of Complementary Slackness.* Suppose that $\mu$ and $\hat{y} = (\hat{u}(x), \hat{\beta})$ are optimal to (P1) and (D1). Then the complementary slackness condition says that

$$T(x,v)\hat{u}(x) + \hat{\beta} = L(x,v), \text{ on } \operatorname{spt}(\mu). \tag{7}$$

If we write $\bar{L} := -\hat{\beta}$, we have

$$\max_{v \in V} \{T(x,v)\hat{u}(x) - L(x,v)\} \le \bar{L}, \tag{8}$$

with equality on $\operatorname{spt}(\sigma)$, where $\sigma := \operatorname{proj}_x \hat{\mu}$.

### 3.2. **Finite Horizon Case.**

3.2.1. *Setup of the Problem.* We consider the measure formulation of the problem directly. Let $\mathbb{T}^n$ and $V$ be the state space and control space respectively. We are looking for a measure $\mu$ on $\Omega \times [0,T] := \mathbb{T}^n \times V \times [0,T]$ to minimize the total cost

$$Cost = \int_0^T \int_\Omega f(x,t,v) d\mu(x,t,v) + \int_\Omega g(x) d\mu|_T(x,v),$$

where $\mu|_T$ is a measure on $\Omega$ with $\mu|_T(Z) := \mu(Z \times \{T\})$, and *running cost* $f$ and *terminal cost* $g$ are given. We need some kind of *flow invariant* condition

$$\int_0^T \int_\Omega \left(\phi_t + L(x,v)\phi\right) d\mu = 0,$$

for all $\phi = \phi(x,t)$ with $\phi(x,0) = \phi(x,T) \equiv 0$ and another condition which generalize the initial condition of the dynamics

$$\int_V \mu|_{t=0}(x,v) = \delta_{x_0},$$

the measure being defined in a similar way as $\mu|_T$. Note that we don't require the measure $\mu$ to be a probability measure.

In summary, the primal problem is

$$\text{(P2)} \qquad \begin{cases} \text{minimize} \displaystyle\int_0^T \int_\Omega f(x,t,v) d\mu(x,t,v) + \int_\Omega g(x) d\mu|_T(x,v), \\ \text{s. t.} \quad \displaystyle\int_0^T \int_\Omega \left(\phi_t + L(x,v)\phi\right) d\mu = 0, \forall \phi \in C_c^\infty([0,T]; C^\infty(\mathbb{T}^n)), \\ \qquad \displaystyle\int_V \mu|_{t=0}(x,v) = \delta_{x_0}, \quad \mu \geq 0. \end{cases}$$

3.2.2. *Dual Problem via Discretization.* The discrete analogue of (P2) is

$$\text{(9)} \qquad \begin{cases} \text{minimize} \displaystyle\sum_{i,j,k} f_{ij}^k \mu_{ij}^k h + \sum_{i,k} g_i \mu_{iJ}^k \\ \text{s. t.} \quad \displaystyle\sum_k \left(\mu_{i,j+1}^k - \mu_{ij}^k - h(L_i^k)^* \mu_{ij}^k\right) = 0, \forall 1 \leq i \leq I, 0 \leq j \leq J-1, \\ \qquad \displaystyle\sum_k \mu_{i,0}^k = \delta_i^{i_0}, \quad \mu_{ij}^k \geq 0, \end{cases}$$

where $i \in \{1, \cdots, I\}, j \in \{0, 1, \cdots, J\}, k \in \{1, \cdots, K\}$ are indices for discretization of $\mathbb{T}^n, [0,T]$ and $V$ respectively and $\delta_\beta^\gamma$ is Kronecker's delta symbol. Note that in the first constraint above, we integrate the corresponding continuum version by parts and approximate $-\partial_t \mu$ by difference quotients with step size $\Delta t = h$. (We also multiplied it by $-h$ to simplify the notation.)

Use the same notations as Section 2,

$$\begin{aligned} &x = \left(\mu_{1,0}^1, \cdots, \mu_{1,J}^1, \mu_{2,0}^1 \cdots, \mu_{2,J}^1, \cdots, \mu_{I,0}^K, \cdots, \mu_{I,J}^K\right); \\ &c = (hf_{1,0}^1, f_{1,1}^1, \cdots, f_{1,J-1}^1, hf_{1,J}^1 + g_1, hf_{2,0}^1, f_{2,1}^1, \cdots, f_{2,J-1}^1, hf_{2,J}^1 + g_2, \cdots \\ &\quad hf_{I,0}^K, f_{I,1}^K, \cdots, f_{I,J-1}^K, hf_{I,J}^K + g_I); \\ &b = (0, \cdots, 0, 1, 0, \cdots, 0), \end{aligned}$$

where 1 lies at the $(IJ + i_0)$-th slot.

We haven't written the matrix $A$ explicitly, but if $y = (u_{ij}, w_i), 1 \le i \le I, 0 \le j \le J-1$, then

$$\begin{aligned}
x \cdot (A^T y) &= Ax \cdot y \\
&= \sum_{i,k} \left[ \sum_{j=0}^{J-1} \left( \mu^k_{i,j+1} - \mu^k_{ij} - h(L^k_i)^* \mu^k_{ij} \right) u_{ij} + \mu^k_{i,0} w_i \right] \\
&= \sum_{i,k} \left[ \mu^k_{i,0}(-u_{i,0} - hL^k_i u_{i,0} + w_i) + \sum_{j=1}^{J-1} \mu^k_{ij}(-u_{ij} + u_{i,j-1} - hL^k_i u_{ij}) + \mu^k_{iJ} u_{i,J-1} \right].
\end{aligned}$$

Then, formally, the dual is

$$\begin{cases}
\text{maximize}\, w_{i_0} \\
\text{s. t.} \quad -u_{i,0} - hL^k_i u_{i,0} + w_i \le f^k_{i,0} h, \quad \forall i, k, \\
\qquad \frac{-u_{ij} + u_{i,j-1}}{h} - L^k_i u_{ij} \le f^k_{ij}, \quad \forall i, k, \forall j = 1, \cdots, J \\
\qquad u_{i,J-1} \le f^k_{iJ} h + g_i, \quad \forall i, k,
\end{cases} \tag{10}$$

where the unknown is $y = (u_{ij}, w_i), 1 \le i \le I, 0 \le j \le J-1$. And we can guess the dual problem of (P2) by letting $h \to 0$,

$$\begin{cases}
\text{maximize} \quad w(x_0), \\
\text{s. t.} \quad -u_t - L(x, v)u \le f(x, t, v), \\
\qquad -u(x, 0) + w(x) \le 0, u(x, T) \le g(x),
\end{cases} \tag{11}$$

with unknown $y = (u(x, t), w(x))$. Clearly, the maximum can only be attained when $w(x) = u(x, 0)$, so the above problem is equivalent to

$$\begin{cases}
\text{maximize} \quad u(x_0, 0), \\
\text{s. t.} \quad u_t + L(x, v)u + f(x, t, v) \ge 0, \\
\qquad u(x, T) \le g(x).
\end{cases} \tag{D2}$$

3.2.3. *Weak Duality.* We can check the weak duality directly. For any feasible $\mu$ and $(u(x, t), v(x))$.

$$\begin{aligned}
&\int_0^T \int_\Omega f(x, t, v) d\mu + \int_\Omega g(x) d\mu|_T(x, v) \\
\ge& -\int_0^T \int_\Omega (u_t + L(x, v)u) d\mu + \int_\Omega u(x, T) d\mu|_T(x, v) \\
=& \int_0^T \int_\Omega u \left(\partial_t \mu - L(x, v)^* \mu\right) - \int_\Omega u(x, t) d\mu \Big|_0^T + \int_\Omega u(x, T) d\mu|_T(x, v) \\
=& u(x_0, 0).
\end{aligned}$$

Therefore, min(P2)≥max(D2).

3.3. **Discounted Infinite Horizon Case.** Suppose the discount factor is $\lambda > 0$ and $\mathbb{T}^n, V$ are state space and control space respectively. We search for a measure $\mu$ on the space $\Omega \times [0, +\infty) = \mathbb{T}^n \times V \times [0, +\infty)$ to minimize the cost

$$\text{Cost} = \int_0^\infty \int_\Omega e^{-\lambda t} f(x, t, v) d\mu,$$

with the same constraints as the finite horizon case. In other words, the primal problem is

$$\text{(P3)} \qquad \begin{cases} \text{minimize} \displaystyle\int_0^\infty \int_\Omega e^{-\lambda t} f(x, t, v) d\mu, \\ \text{s. t.} \quad \displaystyle\int_0^\infty \int_\Omega \left(\phi_t + L(x, v)\phi\right) d\mu = 0, \forall \phi \in C_c^\infty([0, +\infty); C^\infty(\mathbb{T}^n)), \\ \qquad \displaystyle\int_V \mu|_{t=0}(x, v) = \delta_{x_0}, \quad \mu \geq 0. \end{cases}$$

Similar to the finite horizon case, we get the dual problem

$$\text{(D3)} \qquad \begin{cases} \text{maximize} \quad u(x_0, 0), \\ \text{s. t.} \quad u_t + L(x, v)u + f(x, t, v)e^{-\lambda t} \geq 0, \\ \qquad \displaystyle\lim_{t \to \infty} u(x, t) \leq 0. \end{cases}$$

with unknown $u(x, t)$. It's readily to check that the weak duality holds.

3.4. **Optimal Stopping Problems.**

3.4.1. *Setup of the Classical Problem.* Let's first recall the classical framework of optimal stopping problems. Let $U \subset \mathbb{R}^m$ be a bounded, smooth domain. Suppose $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n, \mathbf{B} : \mathbb{R}^n \to M^{n \times m}$ satisfy the usual assumptions, in the sense that, for each $x \in U$, the stochastic differential equation

$$\text{(12)} \qquad \begin{cases} d\mathbf{X} = \mathbf{b}(\mathbf{X})dt + \mathbf{B}(\mathbf{X})d\mathbf{W} \\ \mathbf{X}(0) = x \end{cases}$$

has a unique solution, where $\mathbf{W}$ is standard $m-$dimensional Wiener process.

Let $\tau = \tau_x$ be the exit time from $U$, and $\theta$ be any stopping time with respect to the filtration $\mathcal{F}$. For each $\theta$ we define the expected cost

$$\text{(13)} \qquad J_x[\theta] := E\left(\int_0^{\theta \wedge \tau} f(\mathbf{X}(s))ds + g(\mathbf{X}(\theta \wedge \tau))\right).$$

We are searching for an optimal stopping time $\theta$ to minimize the $J_x[\theta]$.

As before, we relax the problem by looking for some measure that minimizes the "generalized" cost. This has been studied by Cho and Stockbridge [1]. In our case, since the running cost $f$ and terminal cost $g$ are time-independent, we can simplify their result a little. The linear programming formulation for this problem is to

$$\text{(P4)} \qquad \begin{cases} \text{minimize} \int f d\mu + \int g d\nu \\ \text{s.t.} \quad \int \phi d\nu - \int L\phi d\mu = \phi(x), \forall \phi \in \mathcal{D} \\ \qquad \int \nu = 1, \mu \geq 0, \nu \geq 0, \end{cases}$$

where the unknowns are measures $\mu$ and $\nu$ on $U$, $x$ is the initial point of $\mathbf{X}$, and $L$ is the infinitesimal generator of the process $\mathbf{X}(\cdot)$:

$$L\phi(y) = \frac{1}{2}\sum_{i,j} a^{ij}\frac{\partial^2\phi}{\partial x_i \partial x_j} + \sum_i b^i \frac{\partial \phi}{\partial x_i}, \quad a^{ij} = \sum_k b^{ik}b^{jk},$$

for all $\phi \in \mathcal{D}$, the domain of $L$.

Heuristically, the measure $\mu$ is the occupation measure of $\mathbf{X}(\cdot)$:

$$\mu(\Gamma) = E\left(\int_0^{\tau\wedge\theta} \chi_\Gamma(X(s))ds\right),$$

and the measure $\nu$ is the distribution of the random variable $\mathbf{X}(\tau \wedge \theta)$.

3.4.2. *Dual problem and weak duality.* Repeat the process of discretization and finding the dual as before, we get the dual problem

$$\text{(14)} \qquad \begin{cases} \text{maximize}\, u(x) + \beta \\ \text{s.t.} \quad Lu + f \geq 0 \text{ in } U \\ \qquad g(y) \geq u(y) + \beta, \quad \forall y \in U \end{cases}$$

where the unknown is $(u(y), \beta)$. Note that $L\beta = 0$ since $\beta$ is a constant, we can absorb $\beta$ into $u(y)$ and the above problem is equivalent to

$$\text{(D4)} \qquad \begin{cases} \text{maximize}\, u(x) \\ \text{s.t.} \quad Lu + f \geq 0 \text{ in } U \\ \qquad g(y) \geq u(y), \quad \forall y \in U. \end{cases}$$

It's easy to prove the weak duality,

$$\text{(15)} \qquad \int f d\mu + \int g d\nu \geq -\int Lu d\mu + \int u d\nu = u(x).$$

Also note that if we have $(\hat{\mu}, \hat{\nu})$ admissible to (P4) and $\hat{u}$ admissible to (D4), such that

$$\int f d\hat{\mu} + g d\hat{\nu} = \hat{u}(x),$$

then $(\hat{\mu}, \hat{\nu})$ and $\hat{u}$ are solutions. Moreover,

$$\int (Lu + f) d\hat{\mu} = 0, \int (g - \hat{u}) d\hat{\nu} = 0.$$

### 3.5. **Optimal Switching Control Problems.**

3.5.1. *Introduction.*

We consider the following stochastic control problem. Suppose we have a set of *m regimes* (or controls) $K = \{1, 2, \cdots, m\}$. For a given regime $k \in K$, the state process evolves according to the Ito stochastic differential equation

$$d\xi^k(t) = b^k(\xi^k(t))dt + \sigma^k(\xi^k(t))dw(t), \tag{16}$$

where $w$ is a $n-$dimensional standard Brownian motion and $b^k, \sigma^k$ given.

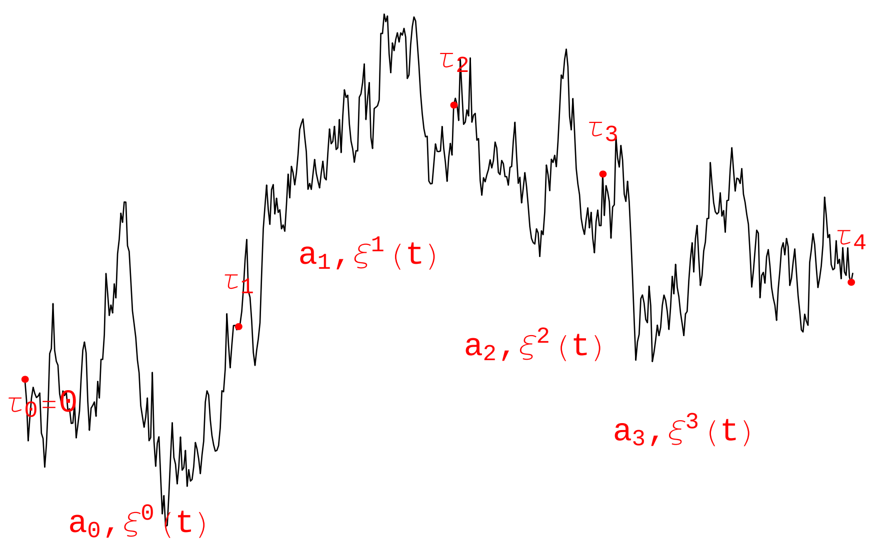
Let $\mathcal{F}_t = \sigma\{w(s) : s \le t\}$ be the history up to time $t$. We shall choose a sequence of regimes $\{a_1, a_2, \cdots\}$ from $K$ and a sequence of $\mathcal{F}_t-$stopping times $\{\tau_1, \tau_2, \cdots\}$ with $\tau_1 < \tau_2 < \cdots$ and $\tau_i \to \infty$ as $i \to \infty$. Suppose the initial state $x_0$ and initial regime $a_0$ are given. To simplify the expressions, denote $\tau_0 = 0$. Now the state process $\xi(\cdot)$ is defined piecewisely,

$$\xi(t) = \xi^i(t), \quad \tau_i \le t \le \tau_{i+1}, \quad i = 0, 1, 2, \cdots,$$

where $\xi^i$ solves

$$d\xi^i = b^{a_i}(\xi)dt + \sigma^{a_i}(\xi)dw, \quad \tau_i \le t \le \tau_{i+1},$$

with initial condition $\xi^i(\tau_i) = \xi^{i-1}(\tau_i)$, for $i \ge 1$ or $\xi^0(0) = x_0$ for $i = 0$.




Let $\mathcal{O} \subset \mathbb{R}^n$ be an open region with smooth boundary. $\eta$ is the exit time of the process from $\mathcal{O}$. For a given policy $\theta = \{\tau_1, a_1, \tau_2, a_2, \cdots\}$, the criteria (total cost) that we want to minimize, among all possible policies, is

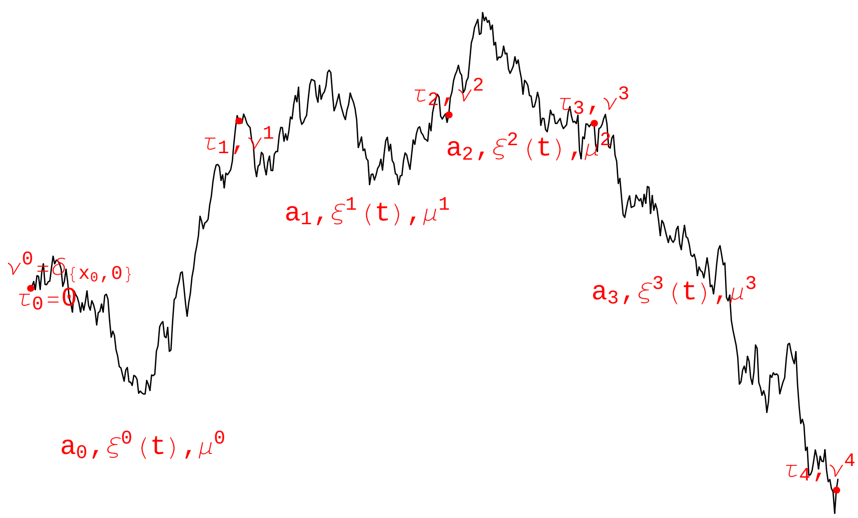
$$J_{x_0,a_0}[\theta] := E\left(\sum_{i=0}^{\infty} \int_{\tau_i}^{\tau_{i+1}} e^{-\alpha t} f^{a_i}(\xi(t)) I_{t<\eta} dt + \sum_{i=1}^{\infty} e^{-\alpha \tau_i} c(a_{i-1}, a_i) I_{\tau_i<\eta}\right),$$

where $\alpha > 0$ is the discount factor, $f^k(x)$ is the running cost when the system is evolving with regime $k$, and $c(k, l)$ is the switching cost from regime $k$ to regime $l$. Note that the indicator functions suggest that we stop the system before the exit time $\eta$.

3.5.2. *Formal Derivation of Linear Programming Formulation.*
Define the occupation measures $\mu^i \in \mathcal{M}(\mathcal{O} \times \mathbb{R}^+)$ by

$$\mu^i(\Gamma) := E\left(\int_{\tau_i}^{\tau_{i+1}} I_\Gamma(\xi(s), s)ds\right), \quad i = 0, 1, 2, \cdots$$

and let the measure $\nu^i \in \mathcal{P}(\mathcal{O} \times \mathbb{R}^+)$ be the joint distribution of $(\xi(\tau_i), \tau_i)$, $i = 0, 1, 2, \cdots$. Note that by definition $\nu^0 = \delta_{(x_0,0)}$ is the Dirac mass.




By a similar argument as in [1], we get

$$J_{x_0,a_0}[\theta] = \sum_{i=0}^{\infty} \int_0^{\infty} \int_{\mathcal{O}} e^{-\alpha t} f^{a_i}(x)\mu^i + \sum_{i=1}^{\infty} \int_0^{\infty} \int_{\mathcal{O}} e^{-\alpha t} c(a_{i-1}, a_i)\nu^i. \tag{17}$$

Moreover, the measures $\mu^i, \nu^i$ satisfy the following "flow-invariant" condition

$$\iint_{\mathcal{O}\times\mathbb{R}^+} \psi\nu^{i+1} - \iint_{\mathcal{O}\times\mathbb{R}^+} \psi\nu^i - \iint_{\mathcal{O}\times\mathbb{R}^+} (\psi_t + L^{a_i}\psi)\mu^i = 0, \quad i = 0, 1, 2, \cdots, \tag{18}$$

for all test functions of the form $\psi = \psi(x, t) = \varphi(x)\beta(t)$ with $\beta(t) \to 0$ as $t \to \infty$, where the operators $L^k (k = 1, \cdots, m)$ are the infinitesimal generators associated to the processes (16), namely

$$L^k\psi = \frac{1}{2}a^k(x) : D_x^2\psi + b^k(x) \cdot D_x\psi, \quad a^k = \sigma\sigma^T.$$

We don't want the undetermined regimes $\{a_1, a_2, \cdots\}$ to appear in the cost functional and conditions, so let's define

$$\tilde{\mu}^k := \sum_{\substack{i\geq 0\\ a_i=k}} \mu^i; \qquad \tilde{\nu}_l^k := \sum_{\substack{i\geq 1, a_i=l\\ a_{i-1}=k}} \nu^i, \quad \forall k, l = 1, 2, \cdots, m; k \neq l. \tag{19}$$

Then the cost functional becomes

$$J_{x_0,a_0} = \sum_{k=1}^{m} \iint_{O\times\mathbb{R}^+} e^{-\alpha t} f^k(x)\tilde{\mu}^k + \sum_{k,l=1}^{m} \iint_{O\times\mathbb{R}^+} e^{-\alpha t} c(k,l)\tilde{\nu}_l^k. \tag{20}$$

If we sum the equations (18) over those $i \geq 1$ with $a_i = k$, we get for all $k = 1, \cdots, m$ and for all test functions $\psi = \varphi(x)\beta(t)$ with $\beta(t)\to\infty$,

$$\sum_{l=1}^{m}\left(\iint_{O\times\mathbb{R}^+} \psi\tilde{\nu}_l^k - \iint_{O\times\mathbb{R}^+} \psi\tilde{\nu}_k^l\right) - \iint_{O\times\mathbb{R}^+} (\psi_t + L^k\psi)\tilde{\mu}^k = \delta_k^{a_0}\psi(x_0, 0). \tag{21}$$

Now we get the linear programming formulation of the problem

$$\begin{cases} \text{minimize } J_{x_0,a_0} = \displaystyle\sum_{k=1}^{m} \iint_{O\times\mathbb{R}^+} e^{-\alpha t} f^k(x)\tilde{\mu}^k + \sum_{k,l=1}^{m} \iint_{O\times\mathbb{R}^+} e^{-\alpha t} c(k,l)\tilde{\nu}_l^k \\ \text{s. t.} \quad \displaystyle\sum_{l=1}^{m}\left(\iint_{O\times\mathbb{R}^+} \psi\tilde{\nu}_l^k - \iint_{O\times\mathbb{R}^+} \psi\tilde{\nu}_k^l\right) - \iint_{O\times\mathbb{R}^+} (\psi_t + L^k\psi)\tilde{\mu}^k = \delta_k^{a_0}\psi(x_0, 0), \forall k, \psi, \\ \qquad \tilde{\mu}^k \geq 0, \tilde{\nu}_l^k \geq 0, \forall k, l \end{cases} \tag{22}$$

the unknowns are $(\tilde{\mu}^k, \tilde{\nu}_l^k), k, l = 1, \cdots, m$.

If we define

$$\bar{\mu}^k(B) = \iint_{O\times\mathbb{R}^+} e^{-\alpha t} I_B(x) d\tilde{\mu}^k(x,t),$$
$$\bar{\nu}_l^k(B) = \iint_{O\times\mathbb{R}^+} e^{-\alpha t} I_B(x) d\tilde{\nu}_l^k(x,t),$$

for any set $B \subset O$, and relax the linear programming problem (22) by requiring that the restriction identity only holds for test functions of the form $\psi(x,t) = e^{-\alpha t}\varphi(x)$, then (22) reduces to

$$\begin{cases} \text{minimize } J_{x_0,a_0} = \displaystyle\sum_{k=1}^{m} \int_O f^k(x)\mu^k + \sum_{k,l=1}^{m} \int_O c(k,l)\nu_l^k \\ \text{s. t.} \quad \displaystyle\sum_{l=1}^{m}\left(\int_O \varphi\nu_l^k - \int_O \varphi\nu_k^l\right) - \int_O (-\alpha\varphi + L^k\varphi)\mu^k = \delta_k^{a_0}\varphi(x_0), \forall k, \varphi, \\ \qquad \mu^k \geq 0, \nu_l^k \geq 0, \forall k, l \end{cases} \tag{P5}$$

here we omit the bar on $\bar{\mu}^k$ and $\bar{\nu}_l^k$ to simplify the notations.

Finally, it's readily to obtain the dual problem

$$\text{(D5)} \qquad \begin{cases} \text{maximize } u^{a_0}(x_0) \\ \text{s. t.} \quad \alpha u^k(x) - L^k u^k(x) \le f^k(x), \\ \qquad u^k(x) - u^l(x) \le c(k,l), \quad \forall 1 \le k, l \le m \end{cases}$$

## References


[1] M. J. Cho and R. H. Stockbridge, *Linear Programming Formulation for Optimal Stopping Problems*, SIAM J. Control Optim., Vol. 40, No. 6, pp. 1965-1982

[2] L. C. Evans, *An Introduction to Stochastic Differential Equations*, Version 1.2, available online.

[3] L. C. Evans and D. Gomes, *Linear Programming Interpretations of Mather's Variational Principle*, Control, Optimization and Calulus of Variations 8 (2002)

[4] Fleming and Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer 1993

[5] Diogo A Gomes, *A Sotchastic Analogue of Aubry–Mather Theory*, Nonlinearity **15**(2002)581-603