REGULARITY OF THE FREE BOUNDARY FOR THE OBSTACLE PROBLEM

1. $(n-1)$-DIMENSION HAUSDORFF MEASURE OF FREE BOUNDARY

Given an bounded open set U, and a function f on U, we try to minimize the energy integral

$$I[w] = \int_U \frac{1}{2}|Dw|^2 - fwdx \tag{1}$$

subject to the condition that

$$w \in H^1(U), \quad w \geq 0.$$

It's well-known that if u is a minimizer, then

$$\begin{cases} -\Delta u \geq f & \text{in } U \\ u \geq 0 & \text{in } U \\ u(-\Delta u - f) = 0 & \text{a.e. in } U, \end{cases} \tag{2}$$

as illustrated in Figure 1.

We define the *coincidence set* $\mathcal{C}$, *non-coincidence set* $\mathcal{N}$ and the *free boundary* Γ as follows,

$$\mathcal{C} = \{x \in U : u(x) = 0\}, \tag{3}$$

$$\mathcal{N} = \{x \in U : u(x) > 0\}, \tag{4}$$

$$\Gamma = \partial\mathcal{N} \cap U. \tag{5}$$

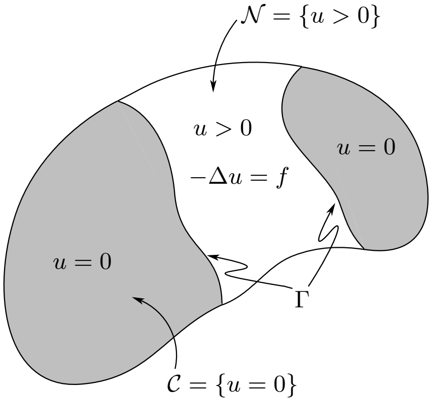

FIGURE 1. Free Boundary

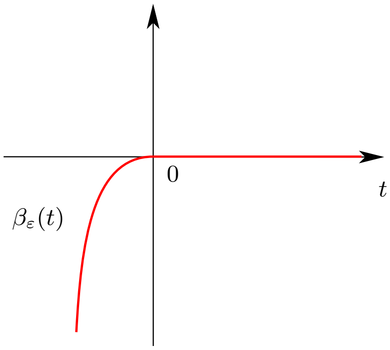

FIGURE 2. Graph of β_ε

Remark 1. *Note that along the free boundary* Γ,

$$u = Du = 0.$$

In stead of writing the obstacle problem in the form of (2), it is convenient to write it as

$$-\Delta u + \beta(u) \ni f, \tag{6}$$

where β is a set valued function defined as

$$\beta(t) = \begin{cases} \{0\}, & t > 0 \\ (-\infty, 0], & t = 0 \\ \emptyset, & t < 0. \end{cases}$$

Now we can approximate (6) with a family of equations,

$$-\Delta u^\varepsilon + \beta_\varepsilon(u^\varepsilon) = f, \tag{7}$$

where β_ε are smooth functions, whose graphs converge to β and satisfy (See Figure 2):

$$\begin{cases} \beta_\varepsilon(t) = 0,\ t \geq 0; \quad \beta_\varepsilon'(t) > 0,\ t < 0; \\ \beta_\varepsilon'' \leq 0; \quad \beta_\varepsilon(t) \to -\infty,\ t < 0. \end{cases}$$

By means of approximation, we have the following theorem

Theorem 1.1. *There exists a constant* $C > 0$, *independent of* ε, *so that*

$$\|u^\varepsilon\|_{W^{2,p}(U)} \leq C;$$
$$\|u^\varepsilon\|_{C^{1,1}(V)} \leq C, \quad \forall V \Subset U;$$

and $u^\varepsilon \to u$, *solution of* (2), *in* $C^1(U)$, *and*

$$\|u\|_{W^{2,p}(U)} \leq C; \quad \|u\|_{C^{1,1}(V)} \leq C, \quad \forall V \Subset U.$$

Proof. Cf. [2]. □

Remark 2. *In general,* $D^2 u \notin C(U)$.

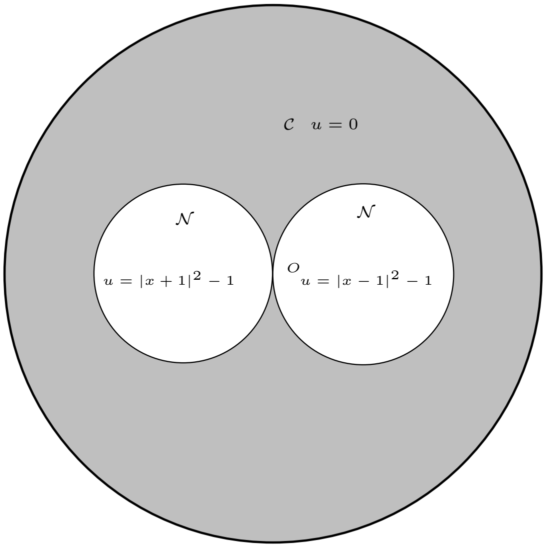

Figure 3. Example of Nonsmooth Free Boundary

Now we start to study the properties of the free boundary Γ. Clearly, Γ need not be smooth. In fact, it may have cusps. For example, as in Figure 3, the free boundary Γ is two tangent circles.

However, we have the following theorem about the smoothness of the free boundary.

Theorem 1.2. *[Caffarelli] If $x_0 \in \Gamma$ satisfies*

$$\lim_{r \to 0} \frac{|B(x_0, r) \cap \mathcal{C}|}{|B(x_0, r)|} \geq \nu > 0, \tag{8}$$

then Γ is smooth near x_0.

For simplicity, we will assume $f \equiv -1$, i.e.,

$$\Delta u = 1, \text{ in } \mathcal{N}. \tag{9}$$

We first show that the $(n-1)$-dimension Hausdorff measure of Γ (away from the boundary of U) is finite, i.e.,

Theorem 1.3. *For any $V \Subset U$, there exists a constant $C > 0$, such that*

$$\mathcal{H}^{n-1}(\Gamma \cap V) \leq C.$$

Before proving the theorem, let's give some lemmas.

Lemma 1.4. *Suppose $x_0 \in \Gamma, B(x_0, r) \subset U$, then there is $y \in \partial B(x_0, r) \cap \mathcal{N}$ such that*

$$u(y) = \max_{\partial B(x_0, r)} u \geq cr^2,$$

where $c = \frac{1}{2n}$.

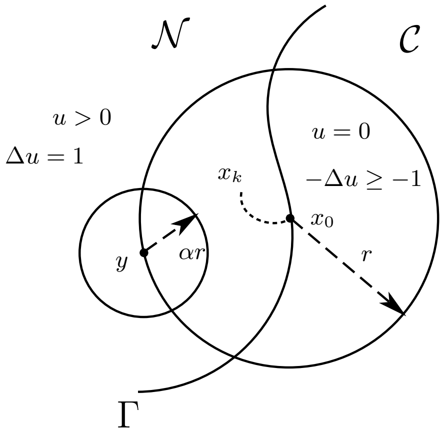

FIGURE 4. Lemma 1.4 & 1.5

Proof. Let's take a sequence $x_k, k = 1, 2, \cdots$, such that $x_k \in \mathcal{N} \cap B(x_0, r)$, and $x_k \to x_0$. Define

$$w_k(x) = u(x) - u(x_k) - \frac{1}{2n}|x - x_k|^2.$$

Then,

$$\begin{aligned} &\Delta w_k(x) = \Delta u(x) - 1 = 0, \ \forall x \in \mathcal{N} \cap B(x_0, r), \text{ by (9)}; \\ &w_k(x_k) = 0; \quad w(x) < 0 \ \forall x \in \Gamma. \end{aligned}$$

By maximum principle, there exists $y_k \in \partial B(x_0, r) \cap \mathcal{N}$, such that

$$w_k(y_k) = \max_{\mathcal{N} \cap B(x_0,r)} w_k.$$

Hence,

$$0 = w_k(x_k) \leq w_k(y_k) = u(y_k) - u(x_k) - \frac{1}{2n}|y_k - x_k|^2.$$

Extract a subsequence of y_k that converge to some $y \in \partial B(x_0, r)$ and let $n \to \infty$ in the above inequality,

$$u(y) \geq \frac{1}{2n} r^2.$$

□

Lemma 1.5. *Suppose $x_0 \in \Gamma$, then*

1. *there exists $\nu > 0$, such that*

$$\frac{|B(x_0, r) \cap \mathcal{N}|}{|B(x_0, r)|} \geq \nu > 0.$$

2. $|\Gamma| = 0$. *(n-dim Lebesgue measure.)*

Proof. According to the previous lemma, there exists $y \in \partial B(x_0, r)$, such that

$$u(y) \geq cr^2.$$

Also, since $u(x_0) = Du(x_0) = 0, |D^2u| \leq C$, we have

$$|Du| \leq Cr, \text{ in } B(x_0, r).$$

Let $z \in B(y, \alpha r)$, where $0 < \alpha < c/C$ fixed, then

$$\begin{aligned} u(z) &\geq u(y) - C\alpha r^2 \\ &\geq cr^2 - C\alpha r^2 > 0. \end{aligned}$$

So $B(y, \alpha r) \subset \mathcal{N}$. Therefore,

$$\frac{|B(x_0, r) \cap \mathcal{N}|}{|B(x_0, r)|} \geq \frac{|B(x_0, r) \cap B(y, \alpha r)|}{|B(x_0, r)|} \geq \frac{C\alpha^n r^n}{r^n} =: \nu > 0.$$

□

Lemma 1.6. *Let* $V \Subset U, V_\delta := \{x \in V | 0 < |Du(x)| < \delta\}$*, then*

$$|V_\delta| \leq C\delta.$$

Proof. Consider the penalizing problem, and differentiate it

$$-\Delta u^\varepsilon + \beta_\varepsilon(u^\varepsilon) = -1 \Rightarrow -\Delta u^\varepsilon_{x_i} + \beta'_\varepsilon(u^\varepsilon)u^\varepsilon_{x_i} = 0$$

Use $\gamma(u^\varepsilon_{x_i})$ as a test function, where γ is defined as $\gamma(t) = t$ for $|t| \leq \delta$, and $\gamma(t) = \delta, t \geq \delta; \gamma(t) = -\delta, t \leq -\delta$. Then

$$\int_V \gamma'(u^\varepsilon_{x_i})u^\varepsilon_{x_i x_k}u^\varepsilon_{x_i x_k} + \underbrace{\beta'_\varepsilon(u^\varepsilon)}_{\geq 0}\underbrace{u^\varepsilon_{x_i}\gamma(u^\varepsilon_{x_i})}_{\geq 0}\,dx = \int_{\partial V} -\frac{\partial u^\varepsilon_{x_i}}{\partial n}\gamma(u^\varepsilon_{x_i})dS \leq C\delta.$$

Note that the last inequality is because the normal derivative is bounded (since $|D^2u^\varepsilon|$ is bounded) by some constant C, and $|\gamma| \leq \delta$. Using the fact that $\gamma' = 1$ on $[-\delta, \delta]$, and sending $\varepsilon \to 0$,

$$\int_{V \cap \{|u_{x_i}| \leq \delta\}} |Du_{x_i}|^2 dx \leq C\delta,$$

hence

$$\int_{V_\delta} |D^2u|^2 \leq C\delta.$$

Finally, note that in V_δ, $|Du| > 0$ and thus

$$1 = |\Delta u|^2 \leq |D^2u|^2.$$

This completes the proof. □

Now let's prove the theorem.

Proof of Theorem 1.3. Fix $r > 0$, by Vitali Covering Lemma, there is an integer N and disjoint balls $\{B(x_i, r)\}_{i=1,\cdots,N}$ such that $x_i \in \Gamma$, and

$$\Gamma \cap V \subset \bigcup_{i=1}^{N} B(x_i, 5r).$$

However, by Lemma 1.5,

$$\begin{aligned}|B(x_i, 5r)| &\le C|B(x_i, r)| \\ &\le C|B(x_i, r) \cap \mathcal{N}| \\ &\le C|B(x_i, r) \cap V_{Cr}|.\end{aligned}$$

The last inequality is because, for any $z \in B(x_i, r) \cap \mathcal{N}$, $0 < |Du(z)| < Cr$ since $Du(x_i) = 0, |D^2u| \le C$. Sum over i and note that the balls $B(x_i, r)$ are disjoint, we have

$$\sum_{i=1}^{n} |B(x_i, 5r)| \le C|V_{Cr}| \le Cr,$$

by Lemma 1.6. So

$$Nr^{n-1} \le C,$$

and C independent of r. Hence

$$\mathcal{H}^{n-1}(\Gamma \cap V) \le C. \qquad \square$$

2. Lipschitz Free Boundary is $C^{1,\alpha}$

In this section we will prove that if the free boundary Γ is locally a Lipschitz graph, then Γ is $C^{1,\alpha}$.

2.1. Flatten the Boundary.

Suppose that Γ is a locally Lipschitz graph, i.e.,

$$\Gamma = \{(x', x_n) | x_n = \gamma(x'), \gamma \text{ is Lipschitz}\}, \tag{10}$$

where $x' = (x_1, \cdots, x_{n-1})$. We make a change of coordinates $\Phi : \mathbb{R}^n \to \mathbb{R}^n$: $x \mapsto y$ defined by

$$\begin{cases} y_i = x_i, \quad i = 1, 2, \cdots, n-1 \\ y_n = x_n - \gamma(x'). \end{cases}$$

Denote by Ψ the inverse transformation of Φ and write

$$\tilde{v}(y) = v(\Psi(y)).$$

Lemma 2.1. *Suppose $\Delta v = 0$ in $\mathcal{N}$ and $v = 0$ in $\mathcal{C}$. then*

1. $(a_{kl}\tilde{v}_{y_k})_{y_l} = 0$ *for* $y_n > 0$, *where* $a_{kl}(y) = \Phi^k_{x_i}(\Psi(y))\Phi^l_{x_i}(\Psi(y))$.
2. $a_{kl}\xi_k\xi_l \ge \nu|\xi|^2$, *for all* $\xi \in \mathbb{R}^n$ *for some* $\nu > 0$.

Note that v will be the slot where we plug in the partial derivatives of the solution u.

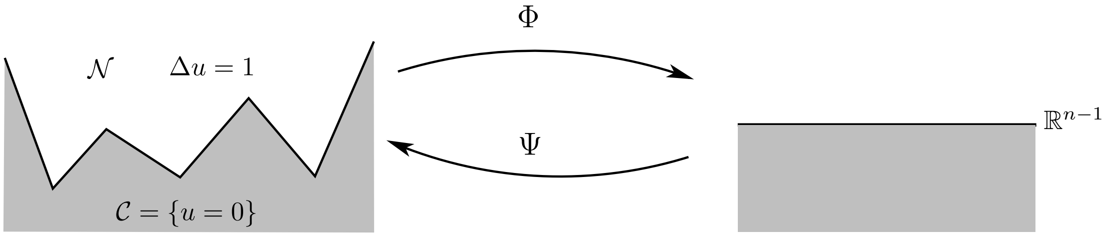

FIGURE 5. Change of Coordinates

Proof. (1) Since

$$\Delta v(x) = \partial_{x_i}\left(\tilde{v}_{y_k}\Phi^k_{x_i}\right) = a_{kl}\tilde{v}_{y_k y_l} + \tilde{v}_{y_k} = 0,$$

we need only to check that $\Phi^k_{x_i x_i} = \partial_{y_l} a_{kl}$.

$$\begin{aligned}
\Phi^k_{x_i x_i} &= -\delta^n_k \Delta\gamma(x'); \\
\partial_{y_l} a_{kl} &= \partial_{y_l}\left(\Phi^k_{x_i}\Phi^l_{x_i}\right) \\
&= \partial_{y_l}\left(\Phi^k_{x_i}\right)\Phi^l_{x_i} + \partial_{y_l}\left(\Phi^l_{x_i}\right)\Phi^k_{x_i} \\
&= \delta^n_k \partial_{y_l}\left(\Phi^n_{x_i}\right)\Phi^l_{x_i} + \underbrace{\partial_{y_n}\left(\Phi^n_{x_i}\right)}_{=0}\Phi^k_{x_i} \\
&= -\delta^n_k \Delta\gamma(x') \quad (l = i \text{ or } \Phi^l_{x_i} = 0).
\end{aligned}$$

(2) We have

$$a_{kl}\xi_k\xi_l = \left(\Phi^k_{x_i}\xi_k\right)\left(\Phi^l_{x_i}\xi_l\right) = |(D\Phi)\xi|^2 \geq \nu|\xi|^2,$$

since $(D\Phi)^{-1}$ exists. Indeed, $\det D\Phi = 1$. □

2.2. **Boundary Harnack Inequalities.** At this point, we need to introduce the so-called Boundary Harnack Inequalities. We define

$$Q^+(z, r) = \{(x', x_n) : |x' - z'| < r, 0 < x_n - z_n < r\},$$

for all $z = (z', z_n) \in \mathbb{R}^n$.

Theorem 2.2. *Suppose* $a_{kl} \in L^\infty, a_{kl} = a_{lk}$, *and* $\lambda|\xi|^2 \leq a_{kl}\xi_k\xi_l \leq \Lambda|\xi|^2$. *Suppose* $u \geq 0$ *is a solution of* $-(a_{kl}u_{x_k})_{x_l} = 0$ *in* $Q^+(0, 2r)$ *with* $u = 0$ *on* $\{x_n = 0\}$. *Then there exists constants* C *only depending on* λ *and* Λ *such that*

$$\max_{Q^+(0,r)} u \leq C u(re_n),$$

where $e_n = (0, \cdots, 0, 1)$.

Proof. Cf. [3, Theorem 1.1]. □

A variant of Theorem 2.2 is the following

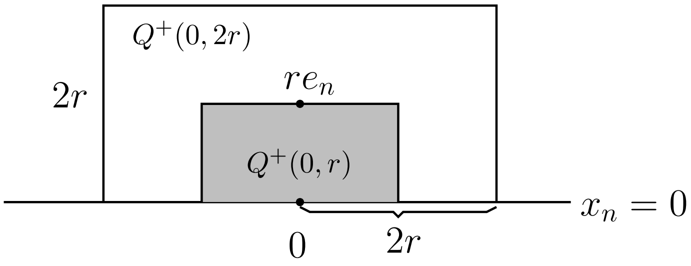

FIGURE 6. Theorem 2.2

Theorem 2.3. *Suppose* $v, w \geq 0$ *are both solutions of*

$$\begin{cases} -(a_{kl}u_{x_k})_{x_l} = 0 & \text{in } Q^+(0,2r) \\ u = 0 & \text{on } \{x_n = 0\} \end{cases}$$

in $Q^+(0,2r)$*, where* $a_{kl} \in L^\infty, a_{kl} = a_{lk}$*, and* $\lambda|\xi|^2 \leq a_{kl}\xi_k\xi_l \leq \Lambda|\xi|^2$*. Also* $w > 0$*. Then there exists constants* C *only depending on* λ *and* Λ *such that*

$$\sup_{Q^+(0,r)} \left(\frac{v}{w}\right) \leq C\frac{v(re_n)}{w(re_n)}.$$

Proof. Cf. [3, Theorem 1.4]. □

We want to apply the theorem to the case when v need not be nonnegative, so we prove

Theorem 2.4. *Assume the same conditions as in Theorem 2.3, but* v *need not be nonnegative. Then there exists* $0 < \alpha < 1$ *such that*

$$\left\|\frac{v}{w}\right\|_{C^{0,\alpha}(Q^+(0,r))} \leq C\frac{\|v\|_{L^\infty}}{w(\frac{3}{2}re_n)}.$$

Proof. (Cf. [1]) Given $z \in \{|x'| < r, x_n = 0\}, R < r$, define

$$M(R) := \sup_{Q^+(z,R)} \left(\frac{v}{w}\right); \quad m(R) := \inf_{Q^+(z,R)} \left(\frac{v}{w}\right);$$
$$\omega(R) := M(R) - m(R).$$

We shall prove that there exists a constant $\eta < 1$ depending only on the ellipticity, such that, in both of the following two cases,

$$\omega(R/2) < \eta\omega(R). \tag{11}$$

Case 1. $\frac{v}{w}(z + \frac{R}{2}e_n) \geq \frac{M(R)+m(R)}{2}.$

Let $v^*(x) = v(x) - m(R)w(x)$ and $w^*(x) = w(x)$, then $v^*(x) \geq 0$ for $x \in Q^+(z, R)$. So by Theorem 2.3,

$$\begin{aligned} \sup_{Q^+(z,R/2)} \frac{w^*}{v^*} &\leq C\frac{w^*}{v^*}(z + \frac{R}{2}e_n) \\ &= \frac{C}{\frac{v}{w}(z + \frac{R}{2}e_n) - m(R)} \text{ (by definition of } v^*, w^*) \\ &\leq \frac{C}{M(R) - m(R)} \text{ (by assumption).} \end{aligned}$$

Therefore,

$$\frac{M(R) - m(R)}{C} \leq \inf_{Q^+(z,R/2)} \left(\frac{v}{w}\right) - m(R) = m(R/2) - m(R)$$

$$\Rightarrow \omega(R/2) = M(R/2) - m(R/2) \leq M(R) - m(R) - \frac{M(R) - m(R)}{C} = \eta\omega(R),$$

where $\eta = 1 - 1/C < 1$ depending only on the ellipticity.

Case 2. $\frac{v}{w}(z + \frac{R}{2}e_n) < \frac{M(R) + m(R)}{2}$.

Repeat the same argument as above, replacing v^* by $v^*(x) = M(R)w(x) - v(x) \geq 0$. We will get the same result (11).

Now that we have (11), a standard trick (Cf., for instance, [6, Page 201, Lemma 8.23]) gives,

$$\omega(R) \leq CR^\alpha \omega(r), \quad \forall R < r.$$

where $\alpha > 0$ depends only on the ellipticity. □

2.3. **Application to Obstacle Problem.** Denote by $v = u_{x_i}, w = u_{x_n}$. Then $\Delta v = \Delta w = 0$ in $\mathcal{N}$ and $v = w = 0$ on Γ. By Lemma 2.1, after changing of coordinates, $\tilde{v}, \tilde{w}$ are both solutions of elliptic equation of divergence form, with $\tilde{v} = \tilde{w} = 0$ on $\{y_n = 0\}$. Therefore, we can apply the previous results.

Theorem 2.5. *Suppose $u_{x_n} > 0$ in $\mathcal{N}$ and the free boundary Γ is locally Lipschitz given by* (10). *Then $\gamma \in C^{1,\alpha}$.*

Proof. Consider the level surfaces $\Gamma_\lambda = \{u = \lambda\}$. Assume Γ_λ is given by $x_n = \gamma_\lambda(x')$.

Claim: There exists $\alpha > 0$, such that $\|\gamma_\lambda\|_{C^{1,\alpha}} \leq C$, independent of λ.

By definition,

$$\begin{aligned} &u(x', \gamma_\lambda(x')) = \lambda \\ \Rightarrow &u_{x_i} + u_{x_n} \cdot (\gamma_\lambda)_{x_i} = 0, \forall i = 1, \cdots, n-1 \\ \Rightarrow &(\gamma_\lambda)_{x_i} = -\frac{u_{x_i}}{u_{x_n}} = -\frac{v}{w}. \end{aligned}$$

Note that

$$\frac{v}{w} = \frac{\tilde{v}}{\tilde{w}}(\Psi(x)) \in C^\alpha,$$

since $\tilde{v}/\tilde{w} \in C^\alpha$ by Theorem 2.4, and Ψ is Lipschitz.

Sending $\lambda \to 0$ and notice that the C^α norm of $D\gamma_\lambda$ is independent of λ, we prove that $\gamma \in C^{1,\alpha}$. □

3. LIPSCHITZ CONTINUITY OF FREE BOUNDARY

In this section, we shall prove that, under some relatively weak conditions, the free boundary Γ is locally Lipschitz and satisfies the conditions of Theorem 2.5, and therefore, $C^{1,\alpha}$.

To this end, we need to study the smoothness of the free boundary of the global solution (blow-up limit) u^∞ defined below.

Suppose $0 \in \Gamma$. Denote by

$$u^r(x) = \frac{1}{r^2} u(rx).$$

Then

$$\|u^r\|_{C^{1,1}} \leq C,$$

and extracting a subsequence $\{u^{r_j}\}$,

$$u^{r_j} \to u^\infty,$$

locally uniformly as $r_j \to 0$.

From now on, we will denote $w = u^\infty$. Then

$$\begin{cases} \Delta w = 1 & \text{in } w > 0, \\ w \geq 0, & \text{in } \mathbb{R}^n, \end{cases} \tag{12}$$

and $0 \in \Gamma_\infty$, where

$$\Gamma_\infty := \partial\{w > 0\}. \tag{13}$$

This is because $\sup\limits_{\overline{B}(0,\delta)} u^r \geq C\delta^2$ and therefore $\sup\limits_{\overline{B}(0,\delta)} w \geq C\delta^2$.

Similarly, we use $\mathcal{C}_\infty$ and $\mathcal{N}_\infty$ to denote the coincidence and non-coincidence set of the solution $w = u^\infty$ in $\mathbb{R}^n$.

3.1. Convexity of Blow-up Limits.

The main result of this section is

Theorem 3.1. *$w : \mathbb{R}^n \to \mathbb{R}$ is convex. In particular, $\mathcal{C}_\infty = \{w = 0\}$ is convex.*

The above theorem follows directly from the following one:

Theorem 3.2. *Let $0 \in \Gamma$. Then there exists a universal modulus of continuity $\sigma(r)$, $\sigma(0^+) = 0$, such that for any unit vector $\xi \in \mathbb{R}^n$,*

$$u_{\xi\xi} \geq -\sigma(|x|).$$

In order to prove Theorem 3.2, we start with the lemmas.

Lemma 3.3. *Let* $x \in \mathcal{N} = \{u > 0\}$*, and* $B(x,r)$ *the largest ball in* $\mathcal{N}$*. Suppose* $\inf_{B(x,r)} u_{\xi\xi} = -\alpha, \alpha > 0$*. Then*

$$u_{\xi\xi}(x) \geq -\alpha + C\alpha^m,$$

where $C > 0, m > 0$ *depending only on dimension.*

Proof. Without loss of generality, assume $r = 1$. Denote by x_0 the point for which $|x_0 - x| = \operatorname{dist}(x, C) = r$. Let $h > 0$ (selected later) and $x_1 \in [x, x_0]$ with $|x_0 - x_1| = h$. Choose $+\xi$ or $-\xi$ so that the direction points inwards to the ball, say ξ. (Note that $D_{\xi\xi} = D_{-\xi,-\xi}$.)

Let $x_2 = x_1 + \frac{\sqrt{h}}{4}\xi \in B(x,1)$, when h small. Thus,

$$\begin{aligned} 0 &\leq u(x_2) \\ &= \underbrace{u(x_1)}_{\leq Ch^2} + \langle \underbrace{\nabla u(x_1)}_{O(h)}, \underbrace{x_2 - x_1}_{O(h^{1/2})} \rangle + \langle \nabla^2 u(\vartheta)(x_2 - x_1), x_2 - x_1 \rangle, \quad \vartheta \in [x_1, x_2] \\ &\leq Ch^2 + Ch^{3/2} + \sup_{[x_1,x_2]} u_{\xi\xi} Ch. \end{aligned}$$

So there exists $x_3 \in [x_1, x_2]$ so that

$$u_{\xi\xi}(x_3) \geq -Ch^{1/2}.$$

If we write $v := u_{\xi\xi} + \alpha$ and choose h s.t. $Ch^{1/2} = \alpha/2$, then

$$\begin{cases} \Delta v = 0, v \geq 0 \text{ in } B(x,1), \\ v(x_3) = u_{\xi\xi}(x_3) + \alpha \geq -Ch^{1/2} + \alpha = \alpha/2. \end{cases}$$

We conclude that (Cf. [5, P86])

$$v(x) \geq \frac{(1 - |x_3|)^{n-1}}{1 + |x_3|} v(x_3) \geq C\alpha^m,$$

since $1 - |x_3| \geq Ch^{1/2} \geq C\alpha, v(x_3) \geq \alpha/2$. □

Based on the above lemma, we have

Lemma 3.4. *Suppose* $0 \in \Gamma, B(0,r) \subset U$*, and* $\inf_{B(0,r)} u_{\xi\xi} \geq -\alpha$*. Then*

$$\inf_{B(0,r/2)} u_{\xi\xi} \geq -\alpha + C\alpha^m.$$

Proof. For any $x \in B(0, r/2) \cap \mathcal{N}$, $r_0 = \operatorname{dist}(x, \mathcal{C}) < r/2$ and hence $\inf_{B(x,r_0)} u_{\xi\xi} \geq \inf_{B(0,r)} u_{\xi\xi} \geq -\alpha$. Applying Lemma 3.3, and using the arbitrariness of x, we get the desired result. □

Proof of Theorem 3.2. Let $r_j = 2^{-j}$ and $\alpha_j = -\inf_{B(0,r_j)} u_{\xi\xi}$. Then by Lemma 3.4, $\alpha_{j+1} \leq \alpha_j - C\alpha_j^m$. Thus,

$$CL\alpha_L^m \leq C \sum_{j=1}^{L} \alpha_j^m \leq \sum_{j=1}^{L} (\alpha_j - \alpha_{j+1}) \Rightarrow \alpha_L \leq (C/L)^{1/m} \to 0, L \to \infty.$$

This completes the proof of the theorem (e.g., $\sigma(r) = C|\log r|^{-1/m}$). □

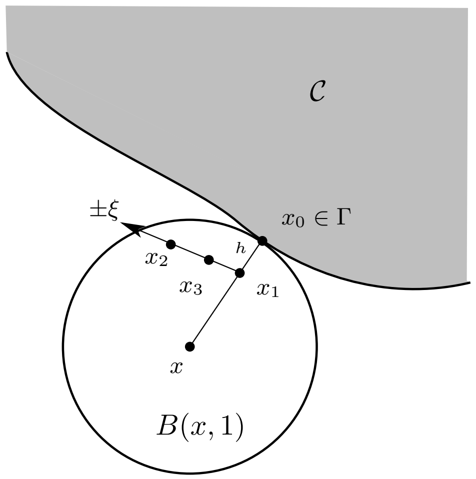

Figure 7. Lemma 3.3

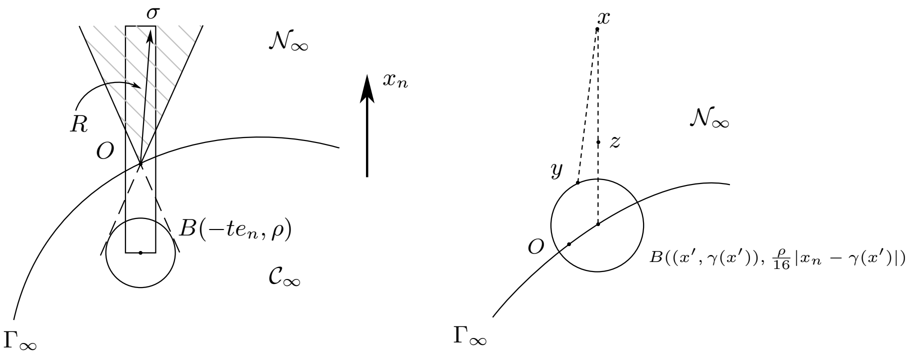

Figure 8. Lemma 3.5

3.2. **Smoothness of Γ_∞.**

Lemma 3.5. $0 \in \Gamma_\infty$. *Suppose* $B(-te_n,\rho) \subset \mathcal{C}_\infty$ *for* $0 < t < \frac{1}{2}, \rho > 0$. *Consider the region* $R = \{(x', x_n) : |x'| < \frac{\rho}{8}, -t < x_n < 1\}$,
(1) $w_\sigma(x) \geq 0$, *for all* $\sigma = (\sigma', \sigma_n), |\sigma| = 1, |\sigma'| \leq \frac{\rho}{8}$.
(2) The level sets $\{w = \lambda\}$ *of* w *are Lipschitz graphs.*

$$x_n = \gamma_\lambda(x'), \|\gamma_\lambda\|_{Lip} \leq C/\rho.$$

(3) $w_{x_n}(x) \geq C(\rho)\operatorname{dist}(x, \Gamma_\infty)$.
(4) $w_\sigma(x) \geq C(\rho)\operatorname{dist}(x, \Gamma_\infty)$ *for* $|\sigma'| < \rho/16$.

Idea of Proof. (1) Consider the shaded cone in Figure 8. Clearly, this cone lies in $\mathcal{N}_\infty = \{w > 0\}$ since $\mathcal{C}_\infty$ is convex (otherwise, $0 \in \mathcal{C}_\infty$). Select σ that lies in this cone and $|\sigma| = 1$, then w_σ is nondecreasing along the direction σ since w is convex. Thus, $w_\sigma(0) \geq 0$. The same argument with 0 (vertex of the cone) replaced by any point x in the region $R \cap \mathcal{N}$ gives the desired result.

(2) The proof of (1) also implies that the level set $\{w = \lambda\}$ (more precisely, the portion of level set in R) can be touched by above with cones of opening angle $\geq C/\rho$, so the level sets (up to the free boundary Γ_∞) are Lipschitz with Lipschitz constant C/ρ.

(3) For $x = (x', x_n) \in R$, we claim that

$$w(x) \geq c\rho^2 |x_n - \gamma(x')|^2.$$

Indeed, consider the ball $B = B((x', \gamma(x')), \frac{\rho}{16}|x_n - \gamma(x')|)$, there is a point $y \in \partial B$ s.t.

$$w(y) \geq c\rho^2 |x_n - \gamma(x')|^2,$$

by Lemma 1.4. By the result of (1), w is nondecreasing along the ray from y to x. (We might need to enlarge R there so that $B \in R$.) Hence the claim.

The claim implies that there exists $z \in [x, (x', \gamma(x'))]$ such that

$$w_{x_n}(z) \geq c\rho^2 |x_n - \gamma(x')|.$$

But this can happen only if $\operatorname{dist}(z, \Gamma_\infty) \geq C\rho^2 |x_n - \gamma(x')|$. By Harnack inequality,

$$w_{x_n}(x) \geq C w_{x_n}(z) \geq C\rho^2 |x_n - \gamma(x')| \geq C(\rho) \operatorname{dist}(x, \Gamma_\infty).$$

(4) follows by expressing such a direction σ as

$$\sigma = a\tilde{\sigma} + b e_n,$$

where $b \geq b_0 > 0, b_0$ a universal constant and $\tilde{\sigma}$ satisfies (1). □

3.3. **Smoothness of Γ.** In order to use the information known about Γ_∞, w to study Γ, u, we prove the following key lemma.

Lemma 3.6. *Let $\Delta h = 0$ in $\mathcal{N} \cap B(0,1)$, $h \geq 0$ on Γ, $h|_{\mathcal{N}_\epsilon} \geq -\epsilon$, and $h|_{\mathcal{N} \setminus \mathcal{N}_\epsilon} \geq 1$, where $\mathcal{N}_\epsilon$ is the ϵ neighborhood of Γ in $\mathcal{N}$. There exists $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$, then*

$$h \geq 0, \text{ in } B(0, \frac{1}{2}) \cap \mathcal{N}.$$

Proof. If not, there exists $x_0 \in B(0, \frac{1}{2}) \cap \mathcal{N}$ with $h(x_0) < 0$, so $x_0 \in B(0, \frac{1}{2}) \cap \mathcal{N}_\epsilon$. Consider

$$v(x) = h(x) - \delta\left(u(x) - \frac{1}{2n}|x - x_0|^2\right),$$

we have

$$\begin{cases} \Delta v = 0 & \text{in } \Omega := B(x_0, \frac{1}{4}) \cap \mathcal{N} \\ v(x_0) = h(x_0) - \delta u(x_0) < 0. \end{cases}$$

Thus $v < 0$ somewhere on $\partial\Omega$ by maximum principle.

(a) Along Γ,

$$v \geq \frac{\delta}{2n}|x - x_0| > 0;$$

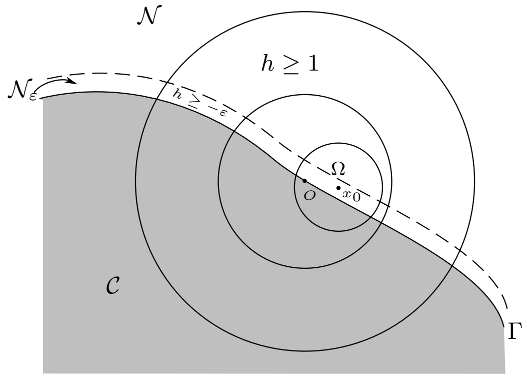

Figure 9. Lemma 3.6

(b) Along $\partial B(x_0, \frac{1}{4}) \cap \mathcal{N}_\epsilon$,

$$v \geq -\epsilon - C\delta\epsilon^2 + \frac{\delta}{2n}\left(\frac{1}{4}\right)^2 > 0,$$

if $\epsilon \geq C\epsilon$, and
(c) Along $\partial B(x_0, \frac{1}{4}) \cap (\mathcal{N} \setminus \mathcal{N}_\epsilon)$,

$$v \geq 1 - C\delta > 0.$$

Contradiction. This proves the lemma. □

Lemma 3.7. *Assume* (8). *Given* $\varepsilon > 0$, *there exists* $r > 0$ *(small) and* $u^{r_j} \to w = u^\infty$ *locally uniformly* ($r_j \to \infty$) *such that*

$$\|u - w\|_{L^\infty(B(0,r))} < \varepsilon, \|\nabla u - \nabla w\|_{L^\infty(B(0,r))} < \varepsilon, \tag{14}$$

and moreover, there is a ball with radius $\rho > 0$ *contained in* $\mathcal{C}_\infty \cap B(0, r)$.

Proof. Take a sequence $r_j \to \infty$ and let $u^{r_j}(x) = \frac{1}{r_j^2}u(r_j x)$. Then $\{u^{r_j}\}_{j=1}^\infty$ is a compact family in $C^{1,\alpha}$ in compact sets, since they all vanish with their gradients at the origin and $C^{1,1}$ in $B(0, r_j/2)$. Extract a subsequence (still denoted by u^{r_j}) that converge locally uniformly to w. Then w is a global solution, i.e., satisfying (12) and $0 \in \Gamma_\infty, w \in C^{1,1}(V)$ for all compact V. So we can take $r > 0$ small enough such that (14) holds.

Because of (8), we can take $\rho > 0$ so small that there is no strip with width $3n\rho$ that covers $B(0, r) \cap \mathcal{C}$. We claim that there's no strip with width $2n\rho$ that covers $B(0, r) \cap \mathcal{C}_\infty$. Suppose the contrary, $B(0, r) \cap \mathcal{C}_\infty$ is contained in a strip $S_{2n\rho}$ with width $2n\rho$. Take the δ-neighborhood of this strip, denoted by $S_{2n\rho+2\delta}$ with ($\delta > 0$ small). In $B(0, r) \setminus S_{2n\rho+2\delta}$ (contained in $\mathcal{N}_\infty$ and away from Γ_∞), $w \geq \alpha > 0$. However, u^{r_j} converge uniformly to w, for large j, $u^{r_j} \geq \alpha/2 > 0$ in $B(0, r) \setminus S_{2n\rho+2\delta}$. Thus

$$B(0, r) \cap \mathcal{C} \subset S_{2n\rho+2\delta},$$

contradiction.

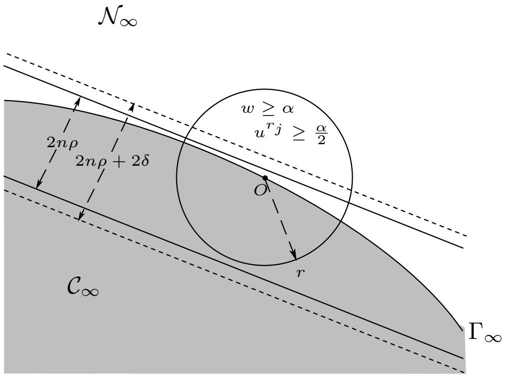

FIGURE 10. Lemma 3.7

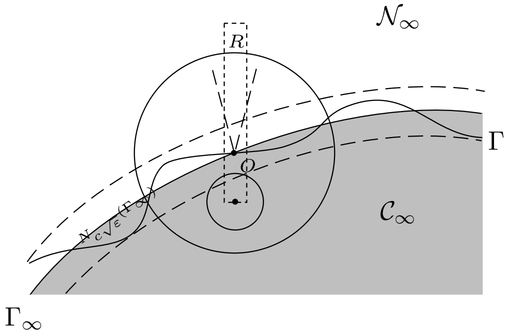

FIGURE 11. Proof of Theorem 1.2

Finally, we apply a lemma of F. John, which says that if E is the largest ellipsoid contained in a convex set, in our case, $B(0,r) \cap \mathcal{C}_\infty$(note that $\mathcal{C}_\infty$ is convex), then $nE \supset B(0,r) \cap \mathcal{C}_\infty$. From the claim, the smallest diameter of the ellipsoid E is at least 2ρ and therefore $B(0,r) \cap \mathcal{C}_\infty$ contains a ball of radius ρ. □

With these lemmas at hand, let's prove the main theorem now.

Proof of Theorem 1.2. First, given $\varepsilon > 0$, take the blow-up limit w as in Lemma 3.7. Then w satisfies the conditions of Lemma 3.5.

In this proof, we restrict ourselves only in the region $R = \{(x', x_n) : |x'| < \frac{\rho}{8}, -t < x_n < 1\}$.

We **claim** that

$$\Gamma \subset N_{c\sqrt{\varepsilon}}(\Gamma_\infty),$$

where $N_\delta(S)$ is the δ-neighborhood of S, and c is a (small) constant depending on ρ. Indeed, (1) if $x_0 \in \mathcal{N}_\infty$ and $\operatorname{dist}(x_0, \Gamma_\infty) \geq c\sqrt{\varepsilon}$, then by Lemma 3.5, $w(x_0) \geq C(\rho)(c\sqrt{\varepsilon})^2 \geq 2\varepsilon$. Therefore, $u(x_0) > 0$ and $x_0 \notin \Gamma$. (2) On the other hand, $x_0 \in \mathcal{C}_\infty$ and $\operatorname{dist}(x_0, \Gamma_\infty) \geq c\sqrt{\varepsilon}$, then $x_0 \notin \Gamma$, otherwise $B(x_0, c\sqrt{\varepsilon}) \subset \mathcal{C}_\infty$, but by non-degeneracy (Lemma 1.4),

$$\sup_{\partial B(x_0, c\sqrt{\varepsilon})} u \geq C(c\sqrt{\varepsilon})^2 \geq 2\varepsilon,$$

and $w \geq \varepsilon$ at the point where above maximum is attained. Contradiction.

As a consequence of the claim, together with Lemma 3.5, for $|\sigma| = 1, |\sigma'|$ small (say, $\leq \rho/32$)

$$w_\sigma(x) \geq C(\rho) \operatorname{dist}(x, \Gamma_\infty) \geq C(\rho)[\operatorname{dist}(x, \Gamma) - c\sqrt{\varepsilon}],$$

hence

$$u_\sigma(x) \geq C(\rho)[\operatorname{dist}(x, \Gamma) - c\sqrt{\varepsilon}] - \varepsilon,$$

since $\|\nabla u - \nabla w\|_\infty \leq \varepsilon$.

Now we want to apply Lemma 3.6 to $h = \varepsilon^{-1/4} u_\sigma$ and $\epsilon = \frac{2\varepsilon^{1/4}}{C(\rho)}$, where we take ε so small that $\frac{2\varepsilon^{1/4}}{C(\rho)} < (\rho/8)\epsilon_0$. We also replace the ball $B(0,1)$ with $B(0, \rho/8)$. Let's check the conditions of Lemma 3.6.

If $\operatorname{dist}(x, \Gamma) \geq \frac{2\varepsilon^{1/4}}{C(\rho)}$, then for ε small

$$h(x) \geq 2 - C(\rho)(c\varepsilon^{1/4} + \varepsilon^{3/4}) \geq 1.$$

If $\operatorname{dist}(x, \Gamma) \leq \frac{2\varepsilon^{1/4}}{C(\rho)}$,

$$h(x) \geq -C\varepsilon^{1/4}.$$

So we have $h(x) \geq 0$ in $B(0, \rho/16)$ by the lemma. Hence $u_\sigma \geq 0$ in this ball and u also satisfies the conclusions (2)-(4) of Lemma 3.5 by repeating the proof. In particular, the free boundary is a Lipschitz graph and $u_{x_n} > 0$, we can now apply Theorem 2.5 to u and get the desired $C^{1,\alpha}$ result. □

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