The Hecke stability method

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Computational issues

The assumption $k \geq 2$ is very common, and for good reasons!

- $S_1(\Gamma_1(N);\mathbb{C})$ is “not cohomological” and cannot be computed directly using modular symbols.
- $\dim S_1(\Gamma_1(N);\mathbb{C}) = ?$

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sage:  CuspForms(Gamma1(59),1)
NotImplementedError.  :(
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There is yet another subtlety to this computation!

Computational issues

Katz modular forms: Let R be a $\mathbb{Z}[\frac{1}{N}]$-module. We define

$$S_k(\Gamma_1(N); R) = H^0(X_1(N)_R, \underline{\omega}^k(-D))$$

where $\underline{\omega}^k(-D)$ needs some (verbose) explanation.
More concretely, well-behaved functions on triples (E, P, ω).

- For $p \nmid N$, the map

$$S_k(\Gamma_1(N); \mathbb{Z}[\tfrac{1}{N}]) \xrightarrow{r_{N,p}} S_k(\Gamma_1(N); \mathbb{F}_p)$$

  fails to be surjective only if $k = 1$. (This is in Katz.)

$\operatorname{coker}(r_{N,p})$ is the space of **ethereal forms** of level N in char. p.
We would like to know more about these spaces.

Theoretical motivation

If $f \in S_1(\Gamma_1(N); \bar{\mathbb{F}}_p)$ is an ethereal newform, then

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$$

could have image "as large as possible" and $\bar{\rho}_f$ is unramified at p (Coleman–Voloch, Edixhoven).

Mestre (1987) gave the first example of an ethereal newform $f \in S_1(\Gamma_1(1429); \mathbb{F}_8)$ by showing $\rho_f(G_{\mathbb{Q}}) = \mathrm{SL}_2(\mathbb{F}_8)$. Buzzard has found examples in odd characteristics.

The characteristics at which ethereal forms occur can be **huge**! For example, there are ethereal forms of level $N = 7 \cdot 347$ in characteristic $935666449040629144864934236346813$.

Hands-on example

Let's compute $S_1(497, \varepsilon_7; R)$ for $R = \mathbb{Z}$ and all $\mathbb{F}_p$, $p \notin \{2, 7, 71\}$.

- Let $\lambda' \in E_1(7, \varepsilon_7; \mathbb{Z})$ be "the" normalized Eisenstein series, and let $\lambda \in E_1(497, \varepsilon_7; \mathbb{Z})$ be given by $\lambda(E, P) = \lambda'(E, 71P)$.
- Define $V(R) = \lambda^{-1} S_2(\Gamma_0(497); R)$. We have

  $$S_1(497, \varepsilon_7; R) \subset V(R) \subset S_1^*(497, \varepsilon_7; R)$$

  $V(\mathbb{F}_p) = V(\mathbb{Z})$ mod p. (This idea courtesy K. Buzzard.)
- The $S_1(497, \varepsilon_7; R)$ is the "holomorphic subspace" of $V(R)$.
- The left and right are Hecke modules. $V(R)$ is **not**!
- **Claim:** $S_1(497, \varepsilon_7; R) = \{\, f \in V(R) : T_2 f \in V(R) \,\}$. For all such R. (To be proven.)

Hands-on example

Claim: For such R, $S_1(497, \varepsilon_7; R) = \{\, f \in V(R) : T_2 f \in V(R) \,\}$.

This reduces computing $S_1(\cdots)$ to linear algebra on q-expansions!

- Compute $\lambda(q)$ and a $\mathbb{Z}$-basis $\{g_1(q), \ldots, g_{47}(q)\}$ for $S_2(497)$. We need these to precision $2 \cdot 97$.
- Let $f_i(q) = \frac{g_i(q)}{\lambda(q)}$. Then $a_n(T_2 f_i) = a_{2n}(f_i) + a_{n/2}(f_i)$.
- Let $M \in \mathrm{Mat}_{94 \times 97}(\mathbb{Z})$ whose upper 47×97 block is $[a_n(f_i)]_{in}$ and whose lower block is $[a_n(T_2 f_i)]_{in}$.
- M has nullity 0, so $S_1(497, \varepsilon; \mathbb{C}) = 0$.
- Computing elementary divisors of M (away from $\{2, 7, 71\}$) we find that M mod p has nullity 2 iff $p = 23$ or $p = 89$.

Answer: $S_1(497, \varepsilon_7; R)$ is trivial except when $R = \mathbb{F}_{23}$ or $R = \mathbb{F}_{89}$.

Proof of the claim

Basically: poles can propagate along isogenies.
Let $V(R) = \lambda^{-1} S_2(\Gamma_0(497); R)$.
We want to show that if $f \in V(R)$ has a pole, then $T_2 f \notin V(R)$.

- Elements of $V(R)$ can have poles **only** at zeros of λ.
- $\lambda(E, P) = 0$ only if $j(E) = 0$. Prove this either with explicit formulas for λ' (following Khuri-Makdisi) or zero-counting.
- E_0 is 2-isogenous to only two curves, itself and E_{54000}. For $p \geq 13$ there is a **unique** 2-isogeny $E_{54000} \to E_0$ over $\bar{\mathbb{F}}_p$.
- If $f \in V(R)$ has a pole, then $T_2 f$ has a pole over E_{54000}, so $T_2 f \notin V(R)$.

Given $N = 7n$, this allows us to compute $S_1(7n, \varepsilon_7; R)$ for $R = \mathbb{Z}$ and all $R = \mathbb{F}_p$ for all $p \nmid 14n$ **simultaneously**.

Hecke stability

More generally, let (N, ε) be a type (with $\varepsilon^2 = 1$ for simplicity). Suppose $\lambda \in E_1(N, \varepsilon^{-1}; R)$ is nonzero at cusps. We would like $\ell \nmid N$ s.t.

$$S_1(N, \varepsilon; R) = \big[\underbrace{\lambda^{-1} S_2(\Gamma_0(N); R)}_{V(R)}\big]^{\langle T_\ell \rangle} \qquad (*)$$

for all "appropriate" R.

- Because any ordinary component of the ℓ-isogeny graph on $X(1)$ is an **infinite** tree (or volcano), one can show that elements of the RHS can have poles only on $X_1(N)^{\mathrm{ss}}$.
- $(*)$ is true when λ has no zeros on $X_1(N)^{\mathrm{ss}}_R$.
- $(*)$ is also true for $R = \mathbb{F}_p$ when p is "large enough."

Attempting to compute $S_1(N, \varepsilon; R)$ for all appropriate R in this way is the Hecke stability method.

Hecke stability

The Hecke stability method can fail:

$$f(q) = \frac{q\prod_{n\geq 1}(1-q^n)^2(1-q^{11n})^2}{1+2\sum_{n\geq 1}\sum_{d\mid n}\varepsilon_{11}(d)q^n} = q + 3q^2 + 5q^4 + 5q^5 + 3q^6 + \\ 2q^7 + q^8 + 3q^9 + q^{10} + 3q^{11} + 3q^{12} + \cdots \pmod 7$$

is stable under $\langle T_5, T_{59}, T_{131}, T_{191}, \ldots\rangle$ but $S_1(11, \varepsilon_{11}; \mathbb{F}_7) = 0$.

- The zeros of the denominator occur above E_{-32^3}.

However, $f(q)^2$ is not a q-expansion of a weight 2 form!

Hecke stability

Theorem: With $V(R)$ as before with appropriate R,

$$S_1(N,\varepsilon;R) = \{\, f \in V(R)^{\langle T_\ell \rangle} : f(q)^2 \in \widetilde{S_2}(N,\varepsilon^2;R)\,\}$$

for **any** $\ell \nmid N$.

Remark. $V(R)^{\langle T_\ell \rangle}$ is easily computed, but the second condition is a pain—it is truly useful for certification of the Hecke stability method only.

Are we guaranteed $S_1(N,\varepsilon;R) = V(R)^A$ for **some** $A \subset \mathbb{T}$?

Implementation

Input: (N, ε, T_ℓ).

1. Compute $S_1(N, \varepsilon; \mathbb{Z}[\frac{1}{N}])$. This computation produces Hecke stability matrices $M_1, \ldots, M_r$ where

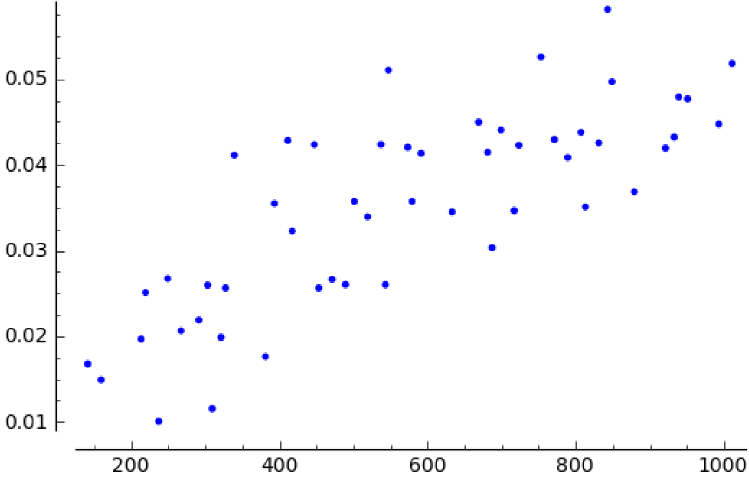
   $$\ker(M_j) \simeq \{\, f \in V(\mathbb{Z}[\tfrac{1}{N}]) : T_\ell f, \ldots, T_\ell^j f \in V(R) \,\}$$

   Terminates when the above space with $j = r$ is $\langle T_\ell \rangle$-stable.
2. Compute and factor the elementary divisors of each M_j. This gives us a finite list L of p to check for ethereal forms.
3. For each $p \in L$, compute $S_1(N, \varepsilon; \mathbb{F}_p)$. Certify these results by squaring q-expansions if required/desired.

Output: $S_1(N, \varepsilon; \mathbb{Z}[\frac{1}{N}])$ and for all $p \nmid 2N$ either $S_1(N, \varepsilon; \mathbb{F}_p)$ or an indication that certification failed over $\mathbb{F}_p$ (**hopefully rare**).

Data

Let $\eta(N,\varepsilon) = \prod_{p \nmid 2N} p^{\dim S_1(N,\varepsilon;\mathbb{F}_p) - \dim S_1(N,\varepsilon;\mathbb{C})}$.



Plot of $\left(3v, \frac{\log \eta(3v,\varepsilon_3)}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(3v)]}\right)$ for v prime.

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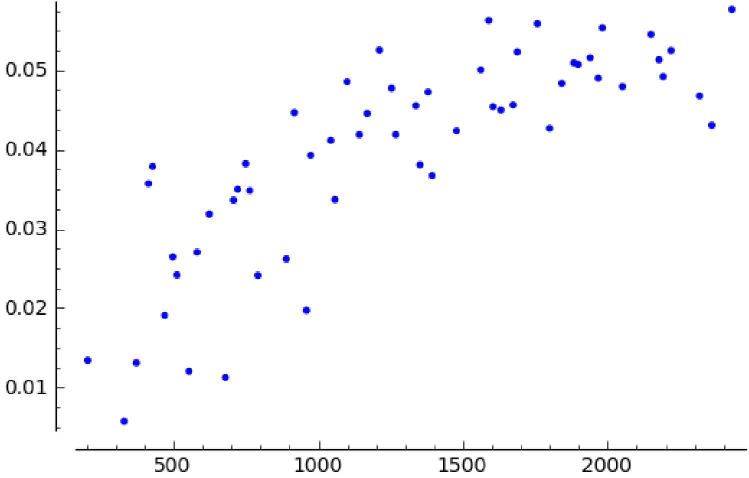

Plot of $\left(7v, \frac{\log \eta(7v,\varepsilon_7)}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(7v)]}\right)$ for v prime.

Data

$\eta(N, \varepsilon)$ is always a square and it tends to have **large** prime factors.

$$\eta(3 \cdot 337) = 47^2 \cdot 5879^2 \cdot 6004682531^2$$
$$\eta(7 \cdot 347) = 79^2 \cdot 935666449040629144864934236346813^2$$

What if we fix a set of prime factors for N?

N	$\eta(N, \varepsilon_3)$
$3 \cdot 5 \cdot 11$	13^2
$3^2 \cdot 5 \cdot 11$	$13^4 \cdot 19^2$
$3 \cdot 5^2 \cdot 11$	$13^6 \cdot 7^4$

Observations/questions arising from these (and other) data:

- There are **multitudes** of ethereal forms. For a fixed ε, data suggest $\eta(N, \varepsilon) \approx C_{\varepsilon}^{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]}$. (Compare with a conjecture of Bergeron–Venkatesh.)
- So far, most give rise to mod p Galois representations with large image (unramified outside N).
- For a fixed S, are there infinitely many p such that there is an ethereal form in characteristic p whose level is a product of primes in S? (What about for $S = \{7\}$, say?)
- Spaces of ethereal forms are always even-dimensional! There is a good reason for this.

Still to be done:

- Organize the data into tables.
  Goal: Have tables of all ethereal newforms with quadratic character and $N \leq 5000$ before I graduate.
- Release nice SAGE and/or Magma code.
- Improve the method for nonquadratic ε.
- Runtime analysis: Computing $S_2(\Gamma_0(N))$ to high precision, taking the elementary divisors of a matrix and factoring them, performing kernel computations...

Thanks!