

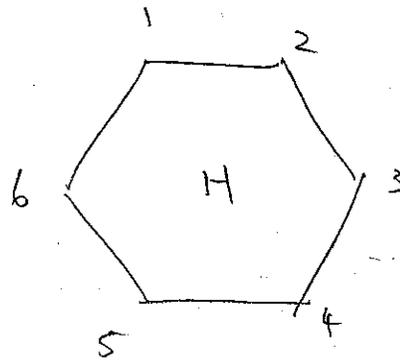
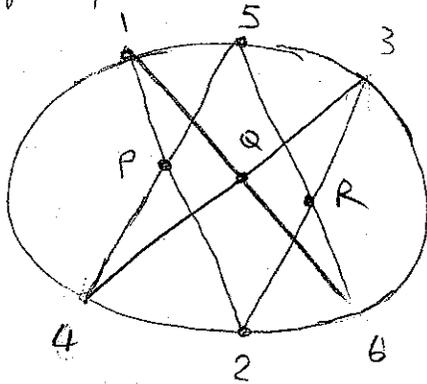
Cayley-Barbarank conjectures (1996, Eisenbud-Green-Harris)

Theorem: (Charles, Cayley-Barbarank)

Let X_1, X_2 be two cubic curves in \mathbb{P}^2 intersecting in 9 pts. Then, if X is a third cubic passing through 8 of those points, then X contains all 9.

Applications:

- Pascal's theorem
- group law on elliptic curves is associative.



If H is a hexagon inscribed in a conic in \mathbb{P}^2 , then opposite sides intersect in 3 collinear points.

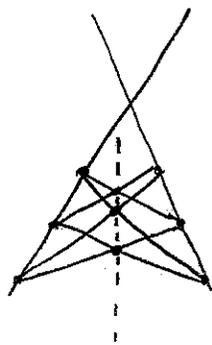
Pf: $X_1 = \overline{12} \cup \overline{34} \cup \overline{56}$

$$X_2 = \overline{23} \cup \overline{45} \cup \overline{16}$$

$$\Rightarrow X_1 \cap X_2 = 6 \text{ vertices of } H \cup \{P, Q, R\}$$

Now $X = C \cup \text{Line}_{PQ} \Rightarrow$ by theorem, $X \ni R$.

ASIDE: Pappas's theorem: (degenerate case) (2)



GOAL: Prove C-B Theorem (X smooth)

1) Cart of characters:

Γ = set of points in \mathbb{P}^n

S = coordinate ring of \mathbb{P}^n , $\mathbb{C}[x_0, \dots, x_n]$

I_Γ = ideal of polys vanishing on Γ

$$S/I_\Gamma = H^0(\Gamma, \mathcal{O}_\Gamma)$$

We denote $(*)_d$ the degree d polynomials inside $*$ ($*$ $\in \{S, I_\Gamma, S/I_\Gamma\}$)

• Define $h_\Gamma(d) \stackrel{\text{def}}{=} \dim_{\mathbb{C}} (S/I_\Gamma)_d$

FACT: h_Γ measures how many independent cond^{ns} the set Γ imposes on forms of degree d .

eg: $\Gamma = 4$ collinear points $\in \mathbb{P}^2$

- $h_\Gamma(d)$?

$$h_\Gamma(1) = \boxed{1}$$

$$h_\Gamma(2) = \boxed{3}$$

$$h_\Gamma(3) = \boxed{4}$$

• CB Theorem: If $\Gamma = X_1 \cap X_2$ (cubics),
then $h_{\Gamma}(3) = 10 - 8$.

2) Divisors, Riemann-Roch

Defⁿ: If C is a smooth plane curve in \mathbb{P}^2 , then
a divisor D is a formal \mathbb{Z} -combⁿ of
points in C

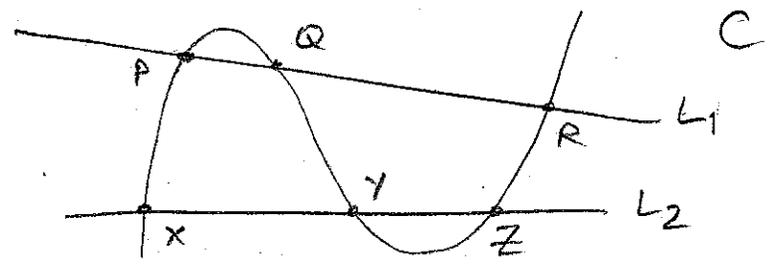
eg: $D = 2P + 3Q - R$

• We say D, D' are linearly equivalent if

$$D - D' = \sum \text{zeros}(f) - \sum \text{poles}(f),$$

for some rational function f on C .

$$\text{deg } D = \sum \text{coeffts.}$$



Then, $(P+Q+R) - (X+Y+Z) = \text{div} \left(\frac{L_1}{L_2} \right)$

↑
call this a hyperplane section, denoted H .

• dH is the divisor of the intersection of a degree d curve with C

• Say D effective if all coefficients are positive.

$$\mapsto |D| = \{ \text{effective divisors lin. equiv. to } D \}$$

can be given the structure of projective linear space.

$l(D) \stackrel{\text{def}}{=} \text{dimension of this space.}$

(4)

Riemann-Roch:

canonical divisor
 $= (d-3)H$

$$l(D) = \text{deg } D - g_C + 1 + l(K_C - D)$$

\uparrow
genus of C
 $= (\text{deg } C - 1)(\text{deg } C - 2)/2$

Propⁿ: X smooth plane curve of degree d ,
 $p \in X$, then every effective divisor
equiv. to $-(d-3)H + p$ actually contains p .

Pf: R-R. \square

More general version of CB:

X_1, X_2 curves of degrees d, e , intersecting in de
points, and C is curve of degree $d+e-3$
contains $de-1$ points. Then, $C \supseteq X_1 \cap X_2$

Pf: (of CB Theorem)

$X_1 = \text{sm. of degree } d$

$X_2 = \text{sm. of degree } e$

$$X_1 \cap X_2 = \{p_1, \dots, p_{de}\}$$

Let C contain $\{p_1, \dots, p_{de-1}\}$, $\text{deg } C = d+e-3$

$$C \cdot X_1 = p_1 + \dots + p_{de-1} + q_1 + \dots + q_{(d-3)d+1}$$

$$K_{X_1} = (d-3)H, \quad C \cdot X_1 \sim (d+e-3)H, \quad p_1 + \dots + p_{de-1} \sim eH$$

$$\rightarrow (d-3)H \sim -p_{de} + q_1 + \dots + q_{d(d-3)+1} \Rightarrow p_{de} \in \{q_1, \dots, q_{d(d-3)+1}\}.$$

\Rightarrow
(R-R,
Thm)