Sample Exam 1. Math 113 Summer 2014.

These problems are practice for the first exam, on group theory. The actual exam will be similar in size and difficulty level, and, like this one, scored out of 125 points.

1. (25 points)

   (a) If G is a group, give the definition of a **normal subgroup** of G.

   (b) If N is a normal subgroup of G, prove that there exists a group H and a homomorphism $f\colon G \to H$ such that $\ker f = N$.

   (c) Let $\mathcal{Q}$ be the group of quaternions, and H the subgroup $\{\pm 1\}$. Prove that $\mathcal{Q}/H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. You may use without proof the fact that up to isomorphism there are just two groups of order 4.

2. (20 points) **True or False**. You don't need to explain your answers.

   (a) If H is a subgroup of G such that $gH = Hg$ for all $g \in G$, then H is contained in the center of G.

   (b) If $f\colon G \to H$ is a homomorphism between groups whose orders are coprime, the $\ker f = G$.

   (c) The permutations $(135)(1462)(65)$ and $(234)(156)$ are conjugate in S_6.

   (d) If $C(g)$ and $C(h)$ are the conjugacy classes of elements $g, h \in G$, $C(g) = C(h)$ if and only if g and h have the same order.

   (e) If H and K are subgroups of a group G which have orders 5 and 6, respectively, then $H \cap K = \{e_G\}$.

   (f) When a group G acts on a set X, the size of any orbit $\mathcal{O}_x$ divides $|X|$.

   (g) If G is a group, and p is a prime dividing $|G|$ such that there are n Sylow p-subgroups of G, then there is a nontrivial homomorphism $G \to S_n$.

   (h) IF G is a group of order 10, generated by two elements, of orders 2 and 5, then $G \cong D_{10}$.

   (i) For a finite group G, and a prime p dividing $|G|$, all p-subgroups of G are conjugate to each other.

   (j) If $f\colon G \to H$ is a homomorphism and $H \cong G/\ker f$, then f is surjective.

3. (30 points) Consider the subgroup $H = \{e, (12)(34), (13)(24), (14)(23)\} \subset S_4$.

   (a) Prove that $H \cong K_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   (b) Prove that H can be written as a union of conjugacy classes in S_4.

   (c) Use (b) to deduce that H is a normal subgroup of S_4.

   (d) Find a group G and a homomorphism from S_4 to G whose kernel is H (this gives an alternate proof that H is normal) [note: do not just use $G = S_4/H$...]

   (e) Determine to which familiar group S_4/H is isomorphic.

4. (25 points)

   (a) Prove that if $\gcd(a, b) = 1$, then $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$ is cyclic.

(b) Prove that if G is a group of order pq, where $p > q$ are primes and $p \not\equiv 1 \bmod q$, then $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$.

(c) Deduce that every group of order 15 is cyclic.

5. (25 points)

   (a) Determine all automorphisms of $\mathbb{Z}/8\mathbb{Z}$.

   (b) Let G be any group. Prove that the following defines an action of G on $\operatorname{Aut} G$: $(g, f) \mapsto g \cdot f$, where $g \cdot f$ is the function on G defined by $g \cdot f(h) = f(g^{-1}hg)$.

   (c) In the case when $G = \mathbb{Z}/8\mathbb{Z}$, prove that the homomorphism $G \to \operatorname{Perm}(\operatorname{Aut} G)$ induced by this action is trivial.

6. (25 points)

   (a) Determine, up to isomorphism, all abelian groups of order 24.

   (b) Let $G = \mathbb{Z}/6Z \times \mathbb{Z}/4\mathbb{Z}$. Find a subgroup H of G such that the quotient group G/H is isomorphic to $K_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   (c) Prove that there is no subgroup H of G such that G/H is isomorphic to $\mathbb{Z}/8\mathbb{Z}$.

   (d) If a group G (no longer necessarily abelian), has 3 Sylow 2-subgroups, prove that G admits a nontrivial homomorphism to S_3 via its action on the set Syl_2 of Sylow 2-subgroups, and use this to conclude that G is not simple.