Some quotient rings. Math 113 Summer 2014.

Claim: $\mathbb{Q}[x]/(x^3-1) \cong \mathbb{R} \times \mathbb{C}$

Let $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. Define

$$f : \mathbb{Q}[x] \to \mathbb{R} \times \mathbb{C}\ ;\ p \mapsto (ev_1(p), ev_\omega(p)),$$

where $\mathrm{ev}_c : \mathbb{Q}[x] \to R\ ;\ p \mapsto p(c) \in R$. Then, f is a homomorphism of rings.

First note that $\ker ev_1 = (x-1)$: indeed, if $p(1) = 0$, then 1 is a root of p and $x-1$ divides p; hence, $\ker ev_1 = (x-1)$. We now claim that $\ker ev_\omega = (1+x+x^2)$: since $\omega^2+\omega+1 = 0$, we have $(x^2+x+1) \subset \ker ev_\omega$. Now, suppose that $p = \sum a_i x^i \in \ker ev_\omega$ so that

$$0 = p(\omega) = \sum a_i \omega^i.$$

As $\omega^3 = 1$ (check!), we find that

$$\omega^k = \begin{cases} 1,\ k \equiv 0 \mod 3, \\ \omega,\ k \equiv 1 \mod 3, \\ \omega^2,\ k \equiv 2 \mod 3. \end{cases}$$

Using that $\omega^2 = \overline{\omega}$, we find (for suitable $a, b, c \in \mathbb{Q}$)

$$0 = p(\omega) = a + b\omega + c\omega^2.$$

Also,

$$p(\omega^2) = p(\overline{\omega}) = \overline{a + b\omega + c\omega^2} = 0$$

Thus, $(x-\omega)(x-\omega^2) = x^2+x+1$ divides p. Hence, $\ker ev_\omega = (x^2+x+1)$.

Now, $\ker f = \ker ev_1 \cap \ker ev_\omega = (x-1) \cap (x^2+x+1) = ((x-1)(x^2+x+1)) = (x^3-1)$.