Homework 4 Math 113 Summer 2014.

Due Wednesday July 16th

Make sure to write your solutions to the following problems in complete English sentences. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make. Problems will be of varying difficulty, and do not appear in any order of difficulty.

1. Let N be a normal subgroup of G, and let $\pi\colon G \to G/N$ be the canonical homomorphism. Let $f\colon G \to H$ be another homomorphism to a group H such that $N \subseteq \ker f$.
   (a) Prove that $\widetilde{f}(gN) = f(g)$ gives a well-defined homomorphism $\widetilde{f}\colon G/N \to H$.
   (b) Prove that $f = \widetilde{f} \circ \pi$.

2. Let G be a p-group, say $|G| = p^r$. Let $1 \le a \le r$. Prove that there exists a subgroup $H \subset G$ such that $|H| = p^a$. (Hint: A proof by induction and HW3, Problem 11a might be useful)

3. (a) Let $f : G \to H$ be a homomorphism of groups, $K \subset H$ a subgroup. Prove that

   $$f^{-1}(K) = \{g \in G \mid f(g) \in K\} \subset G$$

   is a subgroup, and that $\ker f \subset f^{-1}(K)$.
   (b) Let $N \subset G$ be a normal subgroup. Prove that every subgroup of G/N is of the form $\pi_N(H)$, for some subgroup $H \subset G$ containing N.

4. Let $B = \{\text{upper triangular invertible matrices}\}$, a subgroup of $\mathrm{GL}_2(\mathbb{C})$. Consider the following subgroups of B

   $$T = \left\{ \begin{bmatrix} a & \\ & b \end{bmatrix} \mid a, b \in \mathbb{C}^\times \right\}, \quad U = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{C} \right\}.$$

   (a) Prove that $U \subset B$ is normal in B.
   (b) Show that T is not normal in B.
   (c) Prove that $f : T \to B/U\ ;\ t \mapsto tU$ is an isomorphism.
   (d) Explain why B is not isomorphic to $T \times U$.

5. (a) Let $\gcd(a, b) = 1$. Prove that $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$ is cyclic.
   (b) Let G be a finite abelian group, $|G| = p_1 \cdots p_r$, with $p_1, \dots, p_r$ distinct primes. Prove that G is cyclic.
   (c) Give example of two nonabelian groups G_1, G_2 that have order pq and pqr respectively, with p, q, r distinct primes.

6. Determine the list of isomorphism classes of abelian groups of order 360.

7. D_{12} and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ are not isomorphic, but find normal subgroups N_1 of D_{12} and N_2 of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ such that $D_{12}/N_1 \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z})/N_2$. Be sure to justify why the subgroups are normal and why the quotients are isomorphic to each other.