# Homework 2 Math 113 Summer 2014.

*Due Thursday July 3rd*

*Make sure to write your solutions to the following problems in complete English sentences. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make. Problems will be of varying difficulty, and do not appear in any order of difficulty.*

1. Let $g$ be an element of order $k$ in a group $G$.

   a) If $f\colon G \to H$ is a homomorphism, prove that the order of $f(g)$ divides $k$.

   b) If $f\colon G \to H$ is an isomorphism, prove that the order of $f(g)$ is equal to $k$.

2. Find all automorphisms of $\mathbb{Z}/4\mathbb{Z}$.

3. Does $\overline{k} \cdot x = x + k$ define an action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{R}$?

4. Let $U_3(\mathbb{Z}/2\mathbb{Z})$ be the set of matrices

$$U_3(\mathbb{Z}/2\mathbb{Z}) \stackrel{def}{=} \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Z}/2\mathbb{Z} \right\}.$$

   a) Show that $|U_3(\mathbb{Z}/2\mathbb{Z})| = 8$.

   b) Find an element $R$ of order 4 and an element $S$ of order 2 in $U_3(\mathbb{Z}/2\mathbb{Z})$, such that $SRS = R^{-1}$.

   c) Write the group $D_8$ as $\{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, where $r$ is counterclockwise $90^\circ$ rotation of the plane and $s$ is a reflection. Consider the map

$$f : D_8 \to U_3(\mathbb{Z}/2\mathbb{Z})$$

   which sends $s^i r^j$ to $S^i R^j$. This defines a homomorphism (you do not need to prove this). Prove that $f$ is actually an isomorphism.

5. Let $G$ be a finite group, $H \subset G$ a subgroup such that $[G : H] = 2$. Prove that $H$ is normal in $G$.

6. Recall that $W_4$ is isomorphic to $S_4$ (this means that you can do computations involving elements of $S_4$ using the corresponding wiring diagrams).

   a) Show that there exists a subgroup $H \subset S_4$ that is isomorphic to $D_6$. (*This means you need to define an isomorphism* $f : D_6 \to H$)

   b) Show that there exists a subgroup $K \subset S_4$ that is isomorphic to $D_8$. (*This means you need to define an isomorphism* $f : D_8 \to K$. Hint: think about the vertices.)

   c) Is it possible that there exists a subgroup of $S_4$ isomorphic to $D_{10}$? Justify your answer.

7. (Important, and useful for the following exercises!) Prove that "a homomorphism is determined by what it does to the generators", in the following sense. Suppose that a group $G$ is generated by some subset $B$. Suppose $f_1$ and $f_2$ are two homomorphisms to some other group $H$ such that $f_1(g) = f_2(g)$ for all $g$ in $B$. Prove that $f_1 = f_2$. [CAUTION: it does not follow, as is the case in linear algebra, that we can define a homomorphism simply by specifying where to send the generators; one has to be careful about possible relations that the generators may satisfy]

8. Prove that there are no nontrivial homomorphisms[1] from $D_{10}$ to $\mathbb{Z}/5\mathbb{Z}$. [Hint: use problem 1]

9. (CORRECTED) In the dihedral group $D_{12}$ (symmetries of a regulator hexagon centered at the origin with two of its vertices on the $x$-axis) , describe the subgroup $H$ consisting of transformations which fix the line $L$ given by $y = \sqrt{3}x$ (meaning they leave $L$ unchanged). Find the *right*-coset of this subgroup which takes the $x$-axis to $L$. In other words, find an element $g \in D_{12}$ such that the elements of the *right*-coset $Hg$ are all those symmetries which take the $x$-axis to $L$.

10. Using Lagrange's theorem, determine all pairs $m, n$ of positive integers for which there exists a nontrivial homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$.

11. Find all possible actions on the group $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/3\mathbb{Z}$.

12. Define an action of $S_3$ on $\mathbb{R}^3$ as follows: for $\sigma \in S_3$ and $v = (x_1, x_2, x_3) \in \mathbb{R}^3$, set

$$\sigma \cdot v = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

    (a) Show that the subspace $V = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$ has the following property: if $v \in V$, then $\sigma \cdot v$ is also in $V$.

    (b) Can you find a line (one-dimensional subspace) which has this same property?

13. Let $G$ be the set of rotations of the tetrahedron[2], including the identity - it is a subgroup of $T$ (you do not need to check this) of order 12. Prove that $G$ has no subgroup of order 6. [Hints: a) show that such a subgroup would have to contain an element of order 3; b) you can perform the calculations in $W_4$, via problem 5 on HW1]

14. Finish the "converse" part of Lemma 6.1.2 in the notes, namely: let $\alpha$ be a homomorphism $G \to \mathrm{Perm}(S)$; use $\alpha$ to define an action of $G$ on $S$.

[1] The **trivial homomorphism** from $G$ to $H$ is the map $f(g) = e_H$ for all $g \in G$. A homomorphism is **nontrivial** if it is not this one.

[2] The elements of $G$ are listed in the supplement to the solution for HW1.