Homework 1 Math 113 Summer 2014.

Due Thursday June 27th

Make sure to write your solutions to the following problems in complete English sentences. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make. Problems will be of varying difficulty, and do not appear in any order of difficulty.

1. Let C be the set of numbers appearing on a standard clock face. For $a, b \in C$ define

$$a \sim b \iff b = a \text{ } \underline{\text{or}} \text{ } b \text{ appears next to } a \text{ on the clock face.}$$

Is an equivalence relation on C? If so, prove it; if not, explain why.

2. Let $H = \{15a + 9b \mid a, b \in \mathbb{Z}\} \subset \mathbb{Z}$. Find $c \in \mathbb{Z}$ such that $H = \{cr \mid r \in \mathbb{Z}\}$.

3. Prove the following: if $a, b_1, \dots, b_n \in \mathbb{Z}$ such that $\gcd(a, b_i) = 1$, for $i = 1, \dots, n$, then $\gcd(a, b_1 \cdots b_n) = 1$.

4. Prove Lemma 2.2.3: Let A be a nonempty set, $R \subset A \times A$ an equivalence relation on A. If $a, b \in A$ and $b \in [a]$, then $[b] = [a]$.

5. Let W_4 be the set of wiring diagrams on four vertices, T the set of symmetries of the tetrahedron. Recall the definition of **concatenation of wiring diagrams** and **composition of symmetry operations** from Lecture 1.

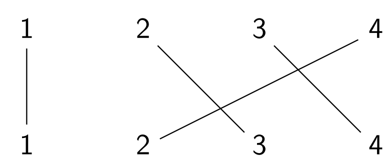
If $A, B \in W_4$, let BA denote the concatenated wiring diagram '*A then B*' (so that A is the top diagram and B is the bottom diagram); if $s, t \in T$ are symmetry operations, let st denote the composed symmetry operation '*t then s*'.

Provide a bijective function[1]

$$f : T \to W_4,$$

such that $f(st) = f(s)f(t) \in W_4$, for any $s, t \in T$ - you only need to check the condition for five distinct pairs of symmetries in T. This means that you must provide a **rule** that assigns to each symmetry operation $s \in T$ exactly one wiring diagram $f(s)$, and in such a way that your assignment obeys the property $f(st) = f(s)f(t)$.

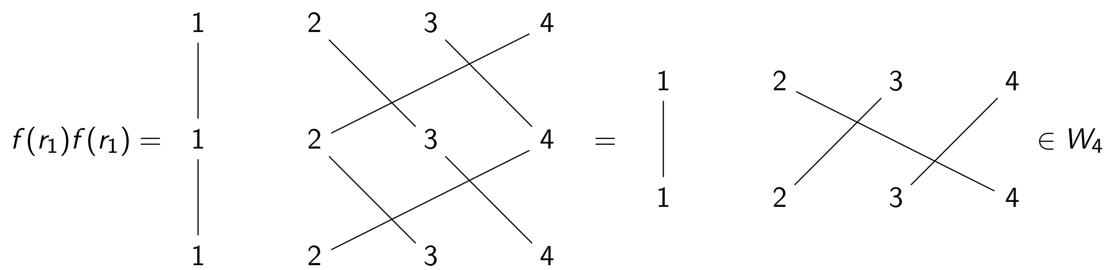
For example, if we label the vertices of the tetrahedron $\{1, 2, 3, 4\}$, and r_1 is the symmetry operation '*rotate the plane spanned by three vertices* $\{2, 3, 4\}$ *by* $2\pi/3$, *keeping vertex* 1 *fixed*', so that vertex 2 is sent to vertex 3, vertex 3 is sent to vertex 4, vertex 4 is sent to vertex 2, then we could let

$f(r_1) =$ 

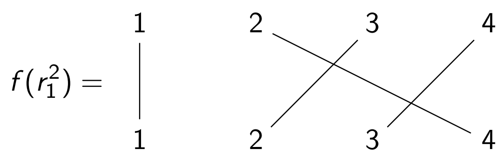
$\in W_4$

Then, if $r_1^2 = r_1 r_1$ is the symmetry operation '*rotate the plane spanned by three vertices* $\{2, 3, 4\}$ *by* $4\pi/3$, *keeping vertex* 1 *fixed*' (ie, do r_1 and then do it again), then the condition given requires

[1]We discussed this example during Lecture 1.



so that



6. Let X be a nonempty set, $X = \bigsqcup_{i=1}^{n} X_i$ a partition of X, for subsets $X_1, \dots, X_n \subset X$. Prove that there exists an equivalence relation R on X such that the equivalence classes of R are precisely the subsets X_i.

7. Let $f : S \to S'$ be a map of sets. Prove that

$$s \sim t \iff f(s) = f(t),$$

defines an equivalence relation on S. In the case of the function $f : \mathbb{R}^2 \to \mathbb{R}\,;\ (x, y) \mapsto x - y$, describe the equivalence classes geometrically.

8. Let X be a nonempty set.

a) Suppose that R_i, $i \in I$, is an arbitrary[2] family of equivalence relations on X. Prove that $\bigcap_{i\in I} R_i$ is an equivalence relation on X.

b) Let $A \subset X \times X$. Show that there exists an equivalence relation R on X such that $R \supset A$.

c) Let $A \subset X \times X$. Prove that there exists an equivalence relation R on X such that $R \supset A$ and so that the following property holds: if R' is an equivalence relation such that $A \subset R'$ then $R \subset R'$.

9. Prove that the cardinality of $S_n = \mathrm{Perm}(\{1, \dots, n\})$ is $n!$. (*Hint: Use induction on n*)

10. Let $z, w \in \mathbb{C}$. Define

$$z \sim w \iff |z| = |w|.$$

Prove that this defines an equivalence relation on $\mathbb{C}$ and describe the equivalence classes geometrically.

[2] If you are worried about infinite families try this problem for a finite family, ie, when I is a finite set.