Facts on S_n. Math 113 Summer 2014.

1. **Elements of S_n are permutations**, i.e., bijective functions from $[n]$ to $[n]$, where $[n]$ is the set $\{1, 2, \dots, n\}$.

2. Permutations can be written as products of **cycles**. A cycle is a permutation of the form (24135), for example. This is a notation for the function which sends

$$
\begin{aligned}
2 &\mapsto 4 \\
4 &\mapsto 1 \\
1 &\mapsto 3 \\
3 &\mapsto 5 \\
5 &\mapsto 2
\end{aligned}
$$

   You read it aloud as "2 goes to 4, 4 goes to 1, 1 goes to 3, 3 goes to 5, 5 goes to 1". Note that the number at the end of the cycle goes back to the first number in the cycle (that's why it's a "cycle").

3. Since the cycles are just notation for functions, they are composed reading **from right to left**. For example, in S_5,

$$
\begin{aligned}
(235)(2154) &= (12)(354) \\
(241)(321) &= (134)(2)
\end{aligned}
$$

   Detailed explanation of the first calculation: open a new cycle by putting "(", and write 1. Then reading from right to left, 1 goes to 5 in the first cycle, and 5 goes to 2 in the second cycle. So overall, 1 goes to 2. We write 2 and then see what happens to 2. 2 goes to 1 in the first cycle, and nothing happens to 1 in the second cycle. So 2 goes to 1, and we close off this cycle (put ")"). Now open a new cycle, starting with 3. Nothing happens to 3 in the first cycle, and 3 goes to 5 in the second. Write 5 and see what happens to it. 5 goes to 4 in the first, and nothing happens to 4 in the second. So write 4. Now, 4 goes to 2, and then 2 goes to 3. Since 3 was the start of this cycle, we close it off. Since we've accounted for what happens to all five numbers, we're done.

   If you do the calculation, and you end up with some 1-cycles, you don't need to write them. For example, in S_6, the permutation (13)5(46)(2) can be written more simply as (13)(46).

4. **Every permutation can be written as a product of disjoint cycles**, meaning no number occurs in more than one cycle. Just do the computation explained above and you will rewrite your permutation in terms of disjoint cycles. In general, it is easier to work with disjoint cycles. One reason is:

5. **Disjoint cycles commute with each other**, even though most of the time permutations do not commute. Another reason is:

6. **The cycle type of a permutation is a list, from highest to lowest, of the lengths of the cycles appearing once you've written it as a product of disjoint cycles**, and we can use cycle type to "classify" the various permutations. Don't forget to include a

1 for each 1-cycle, even though you may have omitted the 1-cycles from your notation. For example, in S_9, the cycle type of (453)(129)(68) is 3,3,2,1 (the last 1 is included for the omitted 1-cycle (7)). Note that you need to know which S_n you're working in to be able to determine any possible missing 1-cycles.

7. **Conjugation is easy in S_n** Recall that any group acts on itself by conjugation, where to conjugate g by h means to turn g into hgh^{-1}. In S_n, If you want to conjugate τ by ω, you just apply the permutation ω to the numbers appearing in τ. For example, in S_6, let $\tau = (124)(36)$, and let $\sigma = (123456)$. Then

$$\sigma\tau\sigma^{-1} = (235)(41)$$

Or if $\tau = (2345)$ and $\sigma = (12)(34)$, we get

$$\sigma\tau\sigma^{-1} = (1435)$$

This seems counterintuitive, because we only apply sigma once and do not have to compute σ^{-1}, but it's a shortcut - we're not actually computing the composition, we're just rewriting τ by renaming the letters occurring there. It's just like changing basis in linear algebra. If I have a linear map whose matrix in the standard basis is A, and I wish to write it in terms of another basis, I take the vectors in the new basis, use them to form a new matrix B, and write the map as BAB^{-1}. Here instead of a basis, we have the numbers $1, \ldots, n$, and σ is "renaming them" in the entries of τ. If you're still skeptical, you can compute the above examples "the long way" and check to see that our trick works.

8. **Conjugacy classes in S_n are determined by cycle type**. The above "conjugation trick" has a powerful theoretical application. Note that in the first example, τ was composed of a 3-cycle and then a 2-cycle, and so was $\sigma\tau\sigma^{-1}$. Similarly in the next example, where both τ and $\sigma\tau\sigma^{-1}$ were 4-cycles. Because of the "conjugation trick", we see that conjugating a permutation does not change its cycle type.

   Note that each cycle type is what's called a **partition of the integer** n, meaning a list of numbers that add up to n. This helps you find all the possible cycle types.

   For example, in S_4, there are 5 conjugacy classes, corresponding to the partitions:

$$\begin{array}{l} 4 \\ 3,1 \\ 2,2 \\ 2,1,1 \\ 1,1,1,1 \end{array}$$