Worksheet 6/30. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework.

1. Let $H = 4\mathbb{Z} = \{4k \mid k \in \mathbb{Z}\}$. Show that H is a subgroup of $\mathbb{Z}$ and determine the cosets of H.

2. Recall that the group $\mathcal{Q}$ of quaternions consists of elements $\pm 1, \pm i, \pm j, \pm k$, with $i^2 = j^2 = k^2 = -1$, and with multiplication of i, j and k defined by

$$ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j.$$

   (a) Find a subgroup of $\mathcal{Q}$ of order 2 and list all of its left cosets.

   (b) Prove that $\mathcal{Q}$ can be generated[1] by the elements i and j.

   (c) Is there a nontrivial homomorphism $f \colon \mathcal{Q} \to \mathbb{Z}/15\mathbb{Z}$?

   (d) (harder, possibly) Is there a homomorphism from $\mathcal{Q} \to D_8$ which sends i to s and j to r (here s is reflection across the x-axis and r a 90 degree counterclockwise rotation)?

3. Find a subgroup H in W_3 and $g \in W_3$ such that $gH \neq Hg$.

4. Let $f \colon G \to G'$ be a homomorphism and set $H = \ker f$. Pick $g \in G$ and set $h = f(g)$. Prove that the coset gH is equal to the pre-image $f^{-1}(\{h\}) = \{a \in G \mid f(a) = h\}$ of h.

5. Let g be an element of a cyclic group G, where $|G| = n$. Prove that an element g^k is a generator of G if and only if k is coprime to n.

6. * Let $G = \mathrm{GL}_2(\mathbb{R})$, $B = \{\text{upper triangular matrices in } G\}$. Show that B is a subgroup of G and

$$G/B = \left\{ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} B \mid a \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} B \right\}.$$

   Show that there is a bijection

$$\{\text{lines in } \mathbb{R}^2\} \longleftrightarrow G/B$$

[1] A group can be **generated by** a set S of elements if all the other elements in the group can be written as products of elements in S.