Worksheet 6/24. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Problems with an asterisk (*) should be more challenging than the rest.

1. Let X be a finite set with $|X| = n$. Prove by mathematical induction that the cardinality of power set $P(X)$ of X is 2^n.

2. Provide a nontrivial example of an equivalence relation on the set of students in this classroom (different from the one given in lecture).

3. Consider the following relation $\sim$ on $\mathbb{R}^n$: $u \sim v$ if and only if $u \cdot v = 0$ (here $\cdot$ means dot product) or $u = av$ for some $a \in \mathbb{R}$. Is this an equivalence relation on $\mathbb{R}^n$? If yes, prove it; if not, provide a counterexample. In the case of $\mathbb{R}^2$, give a geometric explanation for your answer.

4. Let S be the set of 3×3 matrices whose entries are only zeroes and ones, with exactly one 1 in each row and column.

   (a) List all elements of S.

   (b) Is S a group?

   (c) Prove that every element of S can be written as a product of the matrices $S_1 = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$ and $S_2 = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$.

5. * Consider the equivalence relation on $M_2(\mathbb{C})$ defined in example 2.2.6.5 in the notes. Namely, $A \sim B$ if there exists an invertible matrix P such that $A = P^{-1}BP$. Prove that the equivalence class of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is

$$\left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right] = \left\{ A \in M_2(\mathbb{C}) \,|\, A^2 = 0, A \neq 0 \right\}$$