Worksheet July 28. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Problems with an asterisk (*) should be more challenging than the rest. All rings R are assumed commutative and with unity, unless explicitly stated otherwise.

1. Which of the following polynomials are irreducible?
   (a) $x^4 + 4x^2 - 2x + 2 \in \mathbb{Z}[x]$
   (b) $x^6 + x^4 + x^2 + 1 \in \mathbb{Z}[x]$
   (c) $x^6 + x^3 - x^2 - 1 \in \mathbb{Z}[x]$
   (d) $6x^3 + 30x^2 + 210x - 405 \in \mathbb{Q}[x]$ (what about in $\mathbb{Z}[x]$?)
2. Let $f \in \mathbb{Q}[x]$, $\deg f = 2, 3$. Prove that f is irreducible if and only if f does not admit a root.
3. (a) Let k be a field, $p \in k[x]$. Prove that if $\deg f = n$ then f has at most n distinct roots.
   (b) By considering the polynomial $p = x^n - 1$, for suitable n, prove the following: if $G \subset \mathbb{C}^\times$ is a finite subgroup then G is cyclic.
4. Prove that $f = x^p - x - 1 \in \mathbb{Z}/p\mathbb{Z}$ does not admit any roots in $\mathbb{Z}/p\mathbb{Z}$.
5. * This problem determines all automorphisms of $\mathbb{Q}[x]$.
   (a) Let $a, \in \mathbb{Q}$ with $a \neq 0$. Show that $x \mapsto ax + b$ defines a unique automorphism[1] of $\mathbb{Q}[x]$.
   (b) Show that *every* automorphism of $\mathbb{Q}[x]$ is of this form, for some $a, b \in \mathbb{Q}$ with $a \neq 0$.

   In fact, your argument should work for any polynomial ring $R[x]$, as long as R contains no zerodivisors[2].

[1] Recall that an **automorphism** of a ring R is an isomorphism $R \to R$.
[2] Such a ring is called an **integral domain**.