Math 110, Summer 2012: **Practice Exam 1 SOLUTIONS**

Choose 3/5 of the following problems. Make sure to justify all steps in your solutions.

1. Let V be a $\mathbb{K}$-vector space, for some number field $\mathbb{K}$. Let $U \subset V$ be a nonempty subset of V.

i) Define what it means for $U \subset V$ to be a vector subspace of V. Define $\text{span}_{\mathbb{K}} U$.

ii) Prove that $\text{span}_{\mathbb{K}} U$ is a vector subspace of V.

iii) Consider the $\mathbb{Q}$-vector space $V = Mat_2(\mathbb{Q})$ and the subset

$$U = \{I_2, A, A^2\}, \quad \text{where} \quad A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Find $v \in U$ such that

$$\text{span}_{\mathbb{Q}} U = \text{span}_{\mathbb{Q}} U',$$

where $U' = U \setminus \{v\}$. Show that U' is linearly independent.

iv) Find a vector $w \in Mat_2(\mathbb{Q})$ such that $w \notin \text{span}_{\mathbb{Q}} U$. Is the set $U' \cup \{w\}$ linearly independent? Explain your answer.

v) Extend the set $U' \cup \{w\}$ to a basis $\mathcal{B}$ of $Mat_2(\mathbb{Q})$, taking care to explain how you know the set $\mathcal{B}$ you've obtained is a basis.

Solution:

i) $U \subset V$ is a subspace if, for any $u, v \in U, \lambda, \mu \in \mathbb{K}$ we have $\lambda u + \mu v \in U$. We define

$$\text{span}_{\mathbb{K}} U = \{\sum_{j=1}^{k} c_j u_j \in V \mid c_j \in \mathbb{K}, u_j \in U, k \in \mathbb{N}\}.$$

ii) Let $u, v \in \text{span}_{\mathbb{K}} U, \lambda\mu \in \mathbb{K}$. Then, we must have

$$u = c_1 u_1 + ... + c_m u_m, \; v = d_1 v_1 + ... + d_n v_n,$$

where $u_i, v_j \in U, c_i, d_j \in \mathbb{K}$. Then,

$$\lambda u + \mu v = \lambda c_1 u_1 + ... + \lambda c_m u_m + \mu d_1 v_1 + ... + \mu d_n v_n = a_1 w_1 + ... + a_k w_k,$$

where $a_l \in \mathbb{K}, w_l \in U$. Hence, $\lambda u + \mu v \in U$.

iii) We see that

$$U = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \right\}.$$

There is a nontrivial linear relation

$$0_{Mat_2(\mathbb{Q})} = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = 2A - I_2 - A^2.$$

Thus, by the Elimination Lemma we know that

$$\text{span}_{\mathbb{Q}} U = \text{span}_{\mathbb{Q}} \{I_2, A^2\},$$

so that we can take $v = A \in U$. Set $U' = \{I_2, A^2\}$. Suppose that there is a linear relation

$$\lambda_1 I_2 + \lambda_2 A^2 = 0_{Mat_2(\mathbb{Q})} \implies \begin{bmatrix} \lambda_1 + \lambda_2 & 0 \\ -2\lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so that $\lambda_2 = 0, \lambda_1 + \lambda_2 = 0$. Hence, we must have $\lambda_1 = \lambda_2 = 0$ and U' is linearly independent.

iv) If we take

$$w = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then is is easy to see that $w \notin \text{span}_{\mathbb{Q}} U'$ (it's not possible to find $c_1, c_2 \in \mathbb{Q}$ such that $c_1 I_2 + c_2 A^2 = w$). Then, by a result from class (Corollary to Elimination Lemma) we see that $U' \cup \{w\}$ must be linearly independent.

v) Since $U' \cup \{w\}$ consists of 3 vectors and is linearly independent and $Mat_2(\mathbb{Q})$ is has dimension 4, we need only find some vector $z \notin \text{span}_{\mathbb{Q}} U' \cup \{w\}$ to obtain a basis $U' \cup \{w, z\}$ (since this set is then linearly independent, by a similar argument as in iv)). There are many ways to proceed: we will use the most general. Let $\mathcal{S}$ denote the standard ordered basis of $Mat_2(\mathbb{Q})$. Then,

$$[A^2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \ [I_2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ [w]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then, $\underline{b} \in \text{span}_{\mathbb{Q}}\{[A^2]_{\mathcal{S}}, [I_2]_{\mathcal{S}}, [w]_{\mathcal{S}}\}$ if and only if the matrix equation

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \underline{x} = \underline{b},$$

is consistent. So we form the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ -2 & 0 & 0 & b_3 \\ 1 & 1 & 0 & b_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 2 & 0 & b_3 + 2b_1 \\ 0 & 0 & 0 & b_4 - b_1 \end{bmatrix}$$

and there is no pivot in the last column if and only if $b_4 = b_1$. Thus, $\underline{b} \notin \text{span}_{\mathbb{Q}}\{[A^2]_{\mathcal{S}}, [I_2]_{\mathcal{S}}, [w]_{\mathcal{S}}\}$ if and only if $b_4 \neq b_1$. Thus, since the $\mathcal{S}$-coordinate morphism is an isomorphism we have that

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin \text{span}_{\mathbb{Q}} U' \cup \{w\}.$$

Hence, the set

$$\mathcal{B} = \{I_2, A^2, w, e_{11}\},$$

is linearly independent. As this set contains 4 vectors then it must be a basis, by a result from class.

Alternatively, you can just come up with some matrix $B \in Mat_2(\mathbb{Q})$ and show that $B \notin \text{span}_{\mathbb{Q}} U' \cup \{w\}$. In any case, you will need to prove that you know the B you choose is not an element of $\text{span}_{\mathbb{Q}} U' \cup \{w\}$.

2. i) Let $\mathcal{B} = (b_1, \dots, b_n) \subset V$ be an ordered subset of the $\mathbb{K}$-vector space V. Define what it means for $\mathcal{B}$ to be an ordered basis of V using the notions of linear independence AND span.

ii) Consider the maximal linear independence property: *let $E \subset V$ be a linearly independent subset of the $\mathbb{K}$-vector space V. Then, if $E \subset E'$ and E' is linearly independent then $E' = E$.*

Prove that if $\mathcal{B} \subset V$ is a basis (satisfying the definition you gave in 2a)) then $\mathcal{B}$ is maximal linearly independent.

iii) Consider the $\mathbb{Q}$-vector space $\mathbb{Q}^{\{1,2,3\}} = \{f : \{1,2,3\} \to \mathbb{Q}\}$. Let

$$\mathcal{B} = \{f_1, f_2, f_3\} \subset \mathbb{Q}^{\{1,2,3\}},$$

where

$$f_1(1) = 0, f_1(2) = -1, f_1(3) = 1,\ f_2(1) = 0, f_2(2) = 1, f_2(3) = 1,\ f_3(1) = 1, f_3(2) = 1, f_3(3) = 1.$$

Prove that $\mathcal{B}$ is a basis of $\mathbb{Q}^{\{1,2,3\}}$.

iv) Let $\mathcal{B}$ be as defined in 2iii). Define the $\mathcal{B}$-coordinate morphism

$$[-]_{\mathcal{B}} : \mathbb{Q}^{\{1,2,3\}} \to \mathbb{Q}^3,$$

and determine the $\mathcal{B}$-coordinates of $f \in \mathbb{Q}^{\{1,2,3\}}$, where

$$f(1) = 2, f(2) = -1, f(3) = 2.$$

v) Suppose that $\mathcal{C} = (c_1, c_2, c_3) \subset \mathbb{Q}^{\{1,2,3\}}$ is an ordered basis such that the change of coordinate matrix

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Determine $c_1, c_2, c_3 \in \mathbb{Q}^{\{1,2,3\}}$, ie, for each $i \in \{1,2,3\}$, determine $c_1(i), c_2(i), c_3(i)$.

Solution:

i) $\mathcal{B}$ is a basis if it is a linearly independent set and $\mathrm{span}_{\mathbb{K}}\mathcal{B} = V$. Since it is an ordered set it is an ordered basis.

ii) We know that $\mathcal{B}$ is linearly independent, by definition of a basis. So we must show that,

$$\mathcal{B} \subset \mathcal{B}', \text{ with } \mathcal{B}' \text{ linearly indepndent } \implies \mathcal{B}' = \mathcal{B}.$$

Suppose that $\mathcal{B} \neq \mathcal{B}'$. Then, there is some $v \in \mathcal{B}'$ such that $v \notin \mathcal{B}$. Also, we can't have $v = 0_V$ as $\mathcal{B}'$ is linearly independent (any set containing 0_V is linearly dependent). As $\mathrm{span}_{\mathbb{K}}\mathcal{B} = V$ then there are $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ such that

$$v = \lambda_1 b_1 + \dots + \lambda_n b_n \implies v - \sum_{i=1}^{n} \lambda_i b_i = 0_V.$$

As each $b_i \in \mathcal{B} \subset \mathcal{B}'$ then we have obtained a nontrivial linear relation (the coefficient of v is 1) among vectors in $\mathcal{B}'$, contradicting the linear independence of $\mathcal{B}'$. Hence, our initial assumption that $\mathcal{B} \neq \mathcal{B}'$ must be false so that $\mathcal{B}' = \mathcal{B}$.

iii) Consider the standard ordered basis $\mathcal{S} = \{e_1, e_2, e_3\} \subset \mathbb{Q}^{\{1,2,3\}}$. We are going to show that $\mathcal{B}$ is linearly independent by using the $\mathcal{S}$-coordinate isomorphism: we have

$$[f_1]_{\mathcal{S}} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},\ [f_2]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ [f_3]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then, as

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then the set $\{[f_1]_{\mathcal{S}}, [f_2]_{\mathcal{S}}, [f_3]_{\mathcal{S}}\}$ is linearly independent. Since the $\mathcal{S}$-coordinate morphism is an isomorphism we have that $\{f_1, f_2, f_3\}$ is also linearly independent. Hence, as $\mathbb{Q}^{\{1,2,3\}}$ has dimension 3, $\mathcal{B}$ must be a basis.

iv) We use the change of coordinate matrix

$$P_{\mathcal{B}\leftarrow\mathcal{S}} = P_{\mathcal{S}\leftarrow\mathcal{B}}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ -1 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then, we have

$$[f]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{S}}[f]_{\mathcal{S}} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ -1 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \\ 2 \end{bmatrix}.$$

v) We have

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} = [[c_1]_{\mathcal{B}}[c_2]_{\mathcal{B}}[c_3]_{\mathcal{B}}].$$

Hence,

$$c_1 = f_2 - f_3,\ c_2 = f_1 + f_3,\ c_3 = f_1 - f_2 + f_3,$$

so that

$$c_1(1) = -1, c_1(2) = 0, c_1(3) = 0,\ c_2(1) = 1, c_2(2) = 0, c_2(3) = 2,\ c_3(1) = 1, c_3(2) = -1, c_3(3) = 1.$$

3. i) Define the kernel $\ker f$ of a linear morphism $f : V \to W$ and the rank of f, $\mathrm{rank} f$.

ii) Consider the function

$$f : \mathbb{Q}^3 \to \mathbb{Q}^2 \ ; \ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - x_2 + 2x_3 \\ x_1 + x_3 \end{bmatrix}.$$

Explain briefly why f is a linear morphism. What is the rank of f? Justify your answer. Using only the rank of f prove that f is not injective (do not row-reduce!).

iii) Let $\mathcal{S}^{(2)} \subset \mathbb{Q}^2, \mathcal{S}^{(3)} \subset \mathbb{Q}^3$ be the standard ordered bases. Find invertible matrices $P \in \mathrm{GL}_2(\mathbb{Q})$, $Q \in \mathrm{GL}_3(\mathbb{Q})$ such that

$$Q^{-1}[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}} P = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where $r = \mathrm{rank} f$.

iv) Prove or disprove: for the P you obtained in 3iii), a column of P is a basis of $\ker f$.

Solution:

i) We have

$$\ker f = \{v \in V \mid f(v) = 0_W\},$$

and $\mathrm{rank} f = \dim \mathrm{im} f$, where $\mathrm{im} f = \{w \in W \mid \exists v \in V, \text{ such that } f(v) = w\}$.

ii) For every $\underline{x} \in \mathbb{Q}^3$ we have

$$f(\underline{x}) = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \underline{x},$$

so that f must be linear since it is a matrix transformation. The rank of f is 2: we have that f is surjective since there is a pivot in every row of the matrix

$$[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

By the Rank Theorem

$$3 = \dim \mathbb{Q}^3 = \dim \ker f + \text{rank} f \implies \dim \ker f = 1.$$

Thus, $\ker f neq \{0_{\mathbb{Q}^3}\}$ so that f is not injective.

iii) There is more than one approach to this problem via elementary matrices or following the proof of the classification of morphisms theorem. We will follow the latter: first, we find a basis of $\ker f$. This is the same as finding the solution set of the matrix equation

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \underline{x} = \underline{0}.$$

Hence, we have

$$\ker f = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 + x_3 = 0,\ x_2 - x_3 = 0 \right\} = \left\{ \begin{bmatrix} -x \\ x \\ x \end{bmatrix} \mid x \in \mathbb{Q} \right\}.$$

Thus, the set $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of $\ker f$. We extend this to a basis of $\mathbb{Q}^3$: for example,

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

which can be easily show to be a linearly independent set, hence a basis of $\mathbb{Q}^3$. Now consider the matrix

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let

$$Q = [f(e_1)\ f(e_2)] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then, we must have

$$Q^{-1}[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}} P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

v) By construction, we have a column of P is a basis of $\ker f$. If you obtained P, Q using elementary matrices (so that you may not have obtained the same P, Q as I have) then you can see that, since

$$Q^{-1}[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}} P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

the last column of $Q^{-1}[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}}P$ is the zero vector in $\mathbb{Q}^2$. Then, if $P = [p_1\ p_2\ p_3]$, so that i^{th} column of P is p_i, then

$$Q^{-1}[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}}P = \left[Q^{-1}[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}}p_1\ Q^{-1}[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}}p_2\ Q^{-1}[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}}p_3\right],$$

so that

$$\underline{0} = Q^{-1}[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}}p_3 = Q^{-1}[f(p_3)]_{\mathcal{S}^{(2)}}.$$

Since Q is invertible we must have

$$[f(p_3)]_{\mathcal{S}^{(2)}} = \underline{0} \implies f(p_3) = \underline{0},$$

and as $\dim \ker f = 1$ we see that p_3 must be define a basis of $\ker f$.

4. i) Let $A \in Mat_n(\mathbb{C})$. Define what it means for A to be diagonalisable.

ii) Suppose that $P^{-1}AP = D$, with D a diagonal matrix and $P \in \mathrm{GL}_n(\mathbb{C})$. Prove that the columns of P are eigenvectors of A.

For 4iii)-vi) we assume that $A \in Mat_2(\mathbb{C})$, $A^2 = I_2$ and that A is NOT a diagonal matrix.

iii) Show that the only possible eigenvalues of A are $\lambda = 1$ or $\lambda = -1$.

iv) Let $u \in \mathbb{C}^2$ be nonzero. Show that $A(Au + u) = Au + u$.

v) Prove that there always exists some nonzero $w \in \mathbb{C}^2$ such that $Aw \neq -w$. Deduce that $\lambda = 1$ must occur as an eigenvalue of A. Prove that $\lambda = 1$ must also occur as an eigenvalue of A.

vi) Deduce that A is diagonalisable.

Solution:

i) A is diagonalisable if it is similar to a diagonal matrix, ie, if there is $P \in \mathrm{GL}_n(\mathbb{C})$ such that $P^{-1}AP = D$, with D a diagonal matrix.

ii) As $P^{-1}AP = D$ then we see that $AP = PD$. Suppose that

$$D = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}.$$

Now, if $P = [p_1\ \cdots\ p_n]$, so the i^{th} column of P is p_i, then

$$AP = [Ap_1\ \cdots\ Ap_n],$$

and

$$PD = [c_1p_1\ \cdots\ c_np_n] = [Ap_1\ \cdots\ Ap_n].$$

Hence, we have

$$Ap_i = c_ip_i,$$

so that the columns of P are eigenvectors of A.

iii) Suppose that λ is an eigenvalue of A. Then, let v be an eigenvector of A with associated eigenvalue λ (so that $v \neq 0_{\mathbb{C}^2}$). Then,

$$v = I_2v = A^2v = A(Av) = A(\lambda v) = \lambda Av = \lambda^2 v.$$

Hence, as $v \neq 0_{\mathbb{C}^2}$, then we must have $\lambda^2 = 1$ so that λ can be either 1 or -1.

iv) We have
$$A(Au+u) = A^2u + Au = Au + u.$$

v) Suppose that for every nonzero $w \in \mathbb{C}^2$ we have $Aw = -w$. Then, we would have $Ae_1 = -e_1, Ae_2 = -e_2$, implying that $A = -I_2$. However, we have assumed that A is not a diagonal matrix. Hence, there must exist some nonzero $w \in \mathbb{C}^2$ such that $Aw \neq -w$. Now, let $u = Aw + w \neq 0_{\mathbb{C}^2}$. Then, by *iv*) we have
$$Au = u,$$
so that $\lambda = 1$ is an eigenvalue of A. A similar argument shows that $\lambda = -1$ must be an eigenvalue also (here we can find nonzero $z \in \mathbb{C}^2$ such that $Az \neq z$. Then, $A(Az - z) = z - Az$, so that $u' = Az - z$ is an eigenvector with associated eigenvalue $\lambda = -1$).

vi) We have just shown that A admits two distinct eigenvalues. Hence, since A is 2×2 we have that A is diagonalisable (by a result from class).

5. i) Define what it means for a linear endomorphism $f \in \mathrm{End}_{\mathbb{C}}(V)$ to be nilpotent. Define the exponent of f, $\eta(f)$.

ii) Consider the endomorphism
$$f : \mathbb{C}^3 \to \mathbb{C}^3 \; ; \; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 + x_2 - x_3 \\ 0 \end{bmatrix}.$$
Show that f is nilpotent and determine the exponent of f, $\eta(f)$.

iii) Define the height of a vector $v \in \mathbb{C}^3$ (with respect to f), $\mathrm{ht}(v)$. Find a vector $v \in \mathbb{C}^3$ such that $\mathrm{ht}(v) = 2$.

iv) Find a determine a basis $\mathcal{B}$ of $\mathbb{C}^3$ such that
$$[f]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

v) What is the partition of 3 corresponding to the similarity class of f?

Solution:

i) f is nilpotent if there exists some $r \in \mathbb{N}$ such that $f^r = f \circ \cdots \circ f = 0_{\mathrm{End}_{\mathbb{C}}(V)} \in \mathrm{End}_{\mathbb{C}}(V)$ is the zero morphism. The exponent of f is the smallest integer r such that $f^r = 0$ while $f^{r-1} \neq 0$.

ii) You can check that $f^2 = 0 \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$ and, since f is nonzero, the exponent of f is $\eta(f) = 2$.

iii) We define $\mathrm{ht}(v)$ to be the smallest integer r such that $f^r(v) = 0_{\mathbb{C}^3}$, while $f^{r-1}(v) \neq 0_{\mathbb{C}^3}$. The vector $e_1 \in \mathbb{C}^3$ is such that
$$\mathrm{ht}(e_1) = 2,$$
since $f(e_1) \neq 0_{\mathbb{C}^3}$, while $f^2(e_1) = 0_{\mathbb{C}^3}$.

iv) Using the algorithm from the notes you find that $\mathcal{B} = (f(e_1), e_1, e_2 + e_3)$ is an ordered basis such that
$$[f]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
Indeed, we have
$$H_1 = \ker f = E_0 = \mathrm{span}_{\mathbb{C}} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \; H_2 = \mathbb{C}^3.$$

Then,

$$H_2 = H_1 \oplus G_2 = H_1 \oplus \mathrm{span}_{\mathbb{C}}\{e_1\},$$

and, if $S_1 = \{f(e_1)\} = \{e_1 + e_2\}$,

$$H_1 = H_0 \oplus \mathrm{span}_{\mathbb{C}} S_1 \oplus G_1 = \{0\} \oplus \mathrm{span}_{\mathbb{C}} S_1 \oplus \mathrm{span}\{e_2 + e_3\}.$$

Thus, the table you would obtain is

$$\begin{array}{ll} e_1 & \\ f(e_1) & e_2 + e_3 \end{array}$$

v) The partition associated to the nilpotent/similarity class of f is

$$\pi(f) : 12 \leftrightarrow 1 + 2.$$